# CS 610: Dependence Testing

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Sem 2026-27-I

# How to Write Efficient and Scalable Programs?

**Good choice of algorithms and data structures**

Determines the number of operations executed

**Code that the compiler and architecture can effectively optimize**

Determines the number of instructions executed

**Proportion of parallelizable and concurrent code**

Amdahl's law

**Specialize to the target architecture platform**

Memory hierarchy, cache sizes, advanced features like AMX

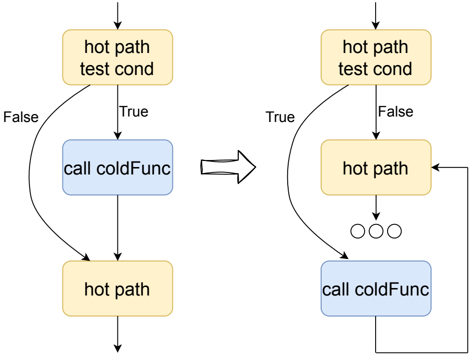
# Role of a Good Parallelizing Compiler

Try and extract performance automatically

## Optimize memory access latency

- Code restructuring optimizations (e.g., loop interchange)
- Prefetching optimizations (e.g., software prefetching)
- Data layout optimizations
- Code layout optimizations

```
// hot path
if (cond)
  coldFunc();
// hot path
```




Denis Bakhvalov. Machine code layout optimizations

# Parallelism Challenges for a Compiler

## On single-core machines

Focus is on register allocation, instruction scheduling, reducing the cost of array accesses

## On parallel machines

- Find **parallelism** in sequential code, find portions of work that can be executed in parallel
- Principle strategy is data decomposition—good idea because data parallelism can scale

# Let us compare the performance!

```
for (i = 0; i < 100000000; i++) {
  W = 1.599999 * X;
  X = 0.999999 * W;
}
```

550–600 ns

```
for (i = 0; i < 100000000; i++) {
  W = 1.599999 * W + 0.000001;
  X = 0.999999 * X;
  Y = 3.14159 * Y + 0.000001;
  Z = Z + 1.0001;
}
```

??? ms

# Can we parallelize the following loops?

**Focus is on loop parallelism because it can provide more savings**

Inter-statement and intra-statement parallelism is limited

```
DO I = 1, 100
  A(I) = A(I) + 1
```

| i | R | W |
|---|---|---|
| 1 | A(1) | A(1) |
| 2 | A(2) | A(2) |
| 3 | A(3) | A(3) |

unroll

```
DO I = 1, 100
  A(I) = A(I-1) + 1
```

| i | R | W |
|---|---|---|
| 1 | A(0) | A(1) |
| 2 | A(1) | A(2) |
| 3 | A(2) | A(3) |

# Data Dependences

```
S1    a = b + c
S2    d = a * 2
S3    a = c + 2
S4    e = d + c + 2
```

**Execution constraints**

- S2 must execute after S1
- S3 must execute after S2
- S3 must execute after S1
- S3 and S4 can execute concurrently (in any order)

There is a **data dependence** from S1 to S2 if and only if

(i) Both statements access the same memory location,

(ii) At least one of the accesses is a write,

(iii) There is a feasible execution path at run-time from S1 to S2.

Control dependence also gives rise to execution constraints, but we will ignore those

# Types of Dependences Based on Memory Accesses

**Flow (a.k.a. true or RAW)**
(denoted by $S_1 \delta S_2$)

```
S1    X = ...
S2    ... = X
```

**Anti (a.k.a. WAR)**
(denoted by $S_1 \delta^{-1} S_2$)

```
S1    ... = X
S2    X = ...
```

**Output (a.k.a. WAW)**
(denoted by $S_1 \delta^{o} S_2$)

```
S1    X = ...
S2    X = ...
```

Only flow dependences are true dependences, anti and output dependences can be removed by renaming.

# Bernstein's Conditions

- Suppose there are two processes $P_1$ and $P_2$
- Let $I_i$ be the set of all input variables for the process $P_i$
- Let $O_i$ be the set of all output variables for the process $P_i$
- $P_1$ and $P_2$ can execute in parallel (denoted by $P_1||P_2$) if and only if
  - $O_1 \cap I_2 = \phi$
  - $O_2 \cap I_1 = \phi$
  - $O_1 \cap O_2 = \phi$

> Two processes can execute in parallel if they are flow-, anti-, and output-independent
>
> - If $P_i||P_j$, does that imply $P_j||P_i$?
> - If $P_i||P_j$ and $P_j||P_k$, does that imply $P_i||P_k$?

A. Bernstein. Analysis of Programs for Parallel Processing. IEEE Transactions on Electronic Computers, 1966.

# Finding Parallelism in Loops—Is it Easy?

Need to check whether two array subscripts access the same memory location

```
    for i = 1 to N
S1    A[i+1] = A[i] + B[i]
```

```
    for i = 1 to N
S1    A[i+4] = A[i] + B[i]
```

- Statement S1 depends on itself in both examples, however, there is a subtle difference in terms of possible parallelism
- Compilers need formalism to analyze dependences and transform loops

# Enumerate All Dependences in Loops

```
      for i = 1 to 50
S1      A[i] = B[i-1] + C[i]
S2      B[i] = A[i+2] + C[i]
```

Unrolling loops helps figure out dependences

| | | | | | |
|---|---|---|---|---|---|
| S1(1) | A[1] | = | B[0] | + | C[1] |
| S2(1) | B[1] | = | A[3] | + | C[1] |
| S1(2) | A[2] | = | B[1] | + | C[2] |
| S2(2) | B[2] | = | A[4] | + | C[2] |
| S1(3) | A[3] | = | B[2] | + | C[3] |
| S2(3) | B[3] | = | A[5] | + | C[3] |

- large loop bounds
- loop bounds may not be known at compile time

# Normalized Iteration Number

Parameterize the statement with the loop iteration number

```
      DO I = 1, N
S1      A(I+1) = A(I) + B(I)


      DO I = L, U, S
S2      ...
```

For a loop where the loop index $I$ runs from $L$ to $U$ in steps of $S$, the normalized iteration number of a specific iteration is $(I - L)/S + 1$, where $I$ is the value of the index on that iteration

# Iteration Vector and Lexicographic Ordering

Given a nest of $n$ loops, the iteration vector $i$ of an iteration of the innermost loop is a vector of integers containing the iteration numbers for each of the loops in order of nesting level.

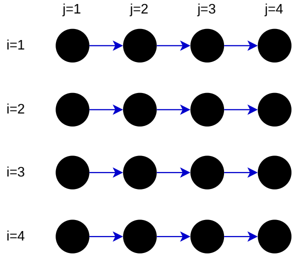
The iteration vector $\vec{i}$ is $\{i_1, i_2, \ldots, i_n\}$ where $i_k$, $1 \leq k \leq n$, represents the iteration number for the loop at nesting level $k$.

- A vector $(d_1, d_2)$ is positive if $(0, 0) < (d_1, d_2)$, i.e., its first non-zero component is positive
- Iteration $\vec{i}$ precedes iteration $\vec{j}$, denoted by $\vec{i} < \vec{j}$, if and only if
  - (i) $i[1 : n-1] < j[1 : n-1]$, or
  - (ii) $i[1 : n-1] = j[1 : n-1]$ and $i_n < j_n$

# Iteration Space Graph

- Represents each dynamic instance of a loop as a point in the graph
- Arrows among points represent dependences

```
      for i = 1 to 4 do
        for j = 1 to 4 do
S1        A(i,j) = A(i,j-1) * x
```




Dimensions of an iteration space depends on the loop nest depth, need not always be rectangular

```
      for i = 1 to 5 do
        for j = i to 5 do
S1        A(i,j) = B(i,j) + C(j)
```

# Loop Dependence

## Loop dependence

There is a **loop dependence** from S1 to S2 in a loop nest iff there exist two iteration vectors $i$ and $j$ such that

(i) $i < j$ or $i = j$ and there is a path from S1 to S2 in the body of the loop,

(ii) S1 accesses memory location $M$ on iteration $i$ and S2 accesses $M$ on iteration $j$, and

(iii) One of these accesses is a write.

- There are two ways in which S2 can depend on S1, where both S1 and S2 are inside a loop
  - **Loop-carried**: S1 and S2 execute in different iterations
  - **Loop-independent**: S1 and S2 execute in the same iteration

```
     DO I = 1, N
S1     A(I+1) = F(I)
S2     F(I+1) = A(I)
```

```
     DO I = 1, N
S1     A(I+1) = F(I)
S2     G(I+1) = A(I+1)
```

# Distance and Direction Vectors

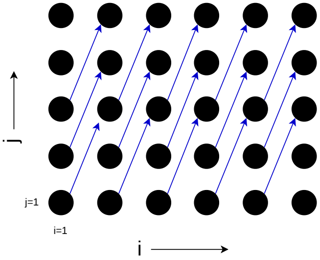
- For each dimension of an iteration space, the **distance** is the number of iterations between accesses to the same memory location
- Dependence distance vector $d(\vec{i}, j)$ is defined as a vector of length $n$ such that $d(\vec{i}, j)_k = j_k - i_k$

```
DO i = 1, 6
  DO j = 1, 5
    A(i,j) = A(i-1,j-2) + 1
```

Distance vector = $(1, 2)$

outer loop

inner loop



# Direction Vectors

Dependence direction vector $D(\vec{i}, j)$ is defined as a vector of length $n$ such that

$$D(i,j)_k = \begin{cases} - \text{ if } D(i,j)_k < 0 \\ 0 \text{ if } D(i,j)_k = 0 \\ + \text{ if } D(i,j)_k > 0 \end{cases}$$

alternate notation

| | |
|---|---|
| $<$ | Positive |
| $>$ | Negative |
| = | Zero |
| * | Mixed |

Distance vector is a more **precise** form of a direction vector

For a valid dependence, the leftmost non-"0" component of the direction vector must be "+"

# Uniform and Non-uniform Distances

A data dependence is non-uniform if the dependence distance is dependent on the iteration vector

```
    for (i = 1; i <= 10; i++) {
      for (j = 1; j <= 10; j++) {
S1      a[i][j] = b[i][j] - e[i+j-2];
S2      c[i][j] = a[i+2][j-1];
S3      a[2*i][j] = d[i][j] + 5;
      }
    }
```

# Summarizing Dependences

```
    DO J = 1, 10
        DO I = 1, 100
S1          A(I,J) = B(I,J) + 5
S2          C(I,J) = A(100-I,J) + 2
```

What are the dependences? How many?

The number of dependences between a pair of accesses is equal to the number of **distinct** direction vectors over all the dependences between those accesses

# Loop-Carried and Loop-Independent Dependences

## Loop-carried

- S1 references location $M$ on iteration $i$
- S2 references M on iteration $j$
- $D(i,j) > 0$ (i.e., contains a "+" as leftmost non-"0" component)

```
    DO I = 1, 10
      DO J = 1, 10
        DO K = 1, 10
S1        A(I,J,K+1) = A(I,J,K)
```

Level of a loop-carried dependence is the leftmost non-"0" index of the dependence D(i,j) (denoted by S1$\delta_l$S2)

## Loop-independent

- S1 refers to location $M$ on iteration $i$
- S2 refers to $M$ on iteration $j$ and $i = j$
- There is a control flow path from S1 to S2 within the iteration

```
    DO I = 1, 9
S1    A(I) = ...
S2    ... = A(10-I)
```

denoted by S1$\delta_\infty$S2

Having a common loop is not necessary

# Distance and Direction Vector Examples

```
    DO I = 1, N
      DO J = 1, M
S1        A(I,J) = ...
S2        ... = A(I,J) + ...
```

```
    DO I = 1, N
      DO J = 1, M
S1        A(I,J) = A(I,1) + ...
```

```
    DO I = 1, N
      DO J = 1, M
        DO K = 1, L
S1        A(I+1,J,K-1) = A(I,J,K) + 10
```

```
   DO I = 1, N
     DO J = 1, M
S1     A(I,J) = A(I,J-3) + A(I-2,J) +
                     A(I-1,J+2) + A(I+1,J-1)
```

# Program Transformations and Validity

- Parallel loop iterations imply potentially random interleaving of statements in the loop body
- Compilers apply loop transformations (e.g., loop permutation) only when it is safe to do so

> A reordering transformation merely changes the relative order of execution of the code, without altering the set of executed statements

- A reordering transformation that preserves every dependence preserves the meaning of the program
- A transformation is valid if, after it is applied, none of the direction vectors for dependences with source and sink in the nest has a leftmost non-“0” component that is “-”

# Utility of Dependence Levels

A reordering transformation preserves **all level-$k$** dependences if it

(i) preserves the iteration order of the level-$k$ loop,

(ii) does not interchange any loop at level $< k$ to a position inside the level-$k$ loop, and

(iii) does not interchange any loop at level $> k$ to a position outside the level-$k$ loop.

```
DO I = 1, 10
  DO J = 1, 10
    DO K = 1, 10
      A(I+1,J+2,K+3) = A(I,J,K) + B
```

```
DO I = 1, 10
  DO K = 10, 1, -1
    DO J = 1, 10
      A(I+1,J+2,K+3) = A(I,J,K) + B
```

Statement order is irrelevant for loop-carried dependences but is important for loop-independent dependences

# Dependence Testing

# Dependence Testing

Dependence testing is used to determine whether dependences exist between two subscripted references to the same array in a loop nest

```
DO I=1, N
  A(4*I) = ...
  ... = A(2*I+2)
```

affine

**Dependence question**

Can $4 * I$ be equal to $2 * I + 2$ for $I \in [1, N]$?

Given (i) two subscript functions $f$ and $g$ and (ii) lower and upper loop bounds $L$ and $U$ respectively, does $f(i_1) = g(i_2)$ have a solution such that $L \le i_1, i_2 \le U$?

# Multiple Loop Indices, Multi-Dimensional Array

Assumptions

- Array subscripts are affine
- Loops are in normalized form

Solving the system of equations for arbitrary functions $f$ and $g$ is undecidable

```
      DO i₁=L₁,U₁,S₁
        DO i₂=L₂,U₂,S₂
          ...
            DO iₙ=Lₙ,Uₙ,Sₙ
S1            X(f₁(i₁,...,iₙ), ..., fₘ(i₁,...,iₙ)) = ...
S2            ... = X(g₁(i₁,...,iₙ), ..., gₘ(i₁,...,iₙ))
```

Let $\alpha$ and $\beta$ be two valid vectors in the iteration space of the loop nest. There is a dependence from S1 to S2 iff

$$\exists \alpha, \beta, \ \ \alpha \leq \beta \wedge f_i(\alpha) == g_i(\beta) \ \ \forall i, 1 \leq i \leq m$$

# Approximate Dependence Testing

The following system of equations with $2n$ variables and $m$ equations is the most common variant

$$
\begin{gathered}
a_{11}i_1 + a_{12}i_2 + \cdots + a_{1n}i_n + c_1 = b_{11}j_1 + b_{12}j_2 + \cdots + b_{1n}j_n + d_1 \\
a_{21}i_1 + a_{22}i_2 + \cdots + a_{2n}i_n + c_2 = b_{21}j_1 + b_{22}j_2 + \cdots + b_{2n}j_n + d_2 \\
\ldots \\
a_{m1}i_1 + a_{m2}i_2 + \cdots + a_{mn}i_n + c_m = b_{m1}j_1 + b_{m2}j_2 + \cdots + b_{mn}j_n + d_m
\end{gathered}
$$

- Solve the system of the form $Ax = B$ for integer solutions
  - A is a $m \times 2n$ matrix and B is a vector of $m$ elements
- Finding solutions to Diophantine equations is NP-complete

# Linear Diophantine Equation

A Linear Diophantine Equation is obtained by equating the two parts of the subscript of an array reference pair

- Coefficients of the loop indices are integers in Diophantine equations

```
    for (i = 1; i <= 10; i++) {
      for (j = 1; j <= 10; j++) {
S1      x[i][j] = 1 + e[i+j-2];
S2      e[i+j] = y[i][j] - 2;
      }
    }
```

For dependence to exist there must exist equality between source iteration instance (S1, $x_1$, $x_2$) and sink iteration instance (S2, $y_1$, $y_2$). The corresponding LDE is

$$x_1 + x_2 - 2 = y_1 + y_2 \quad \text{where} \quad 1 \leq x_1, y_1, x_2, y_2 \leq 10$$

# Conservative and Exact Tests

## Conservative test

- Tests are meant to be cheap at the cost of precision (e.g., GCD test, Banerjee test)
  - Overapproximates dependences (i.e., no false negatives but may have false positives)
  - + Always correct when it determines there is no dependence
  - — May be wrong when it determines there is a dependence

## Exact test

- Detects a dependence if and only if it actually exists
  - Neither false negatives nor false positives
- Computationally more expensive because they require solving integer linear equations (e.g., Omega test)

# Classifying Subscripts

- A subscript is a pair of subscript positions in a pair of array references
  - $A(i, j) = A(i, k) + C$
  - $\langle i, i \rangle$ is the first subscript, $\langle j, k \rangle$ is the second subscript

- A subscript is said to be
  - Zero index variable (ZIV) if it contains no index variable
  - Single index variable (SIV) if it contains only one index variable
    - Strong SIV subscript has same coefficients of index variables (e.g., $i + 1$ and $i$)
    - Weak SIV subscript has different coefficients of index variables (e.g., $i + 1$ and $2i$)
  - Multi index variable (MIV) if it contains more than one index variable

- Consider $A(5, i + 1, j + k) = A(1, i, j + k - 2) + C$
  - First subscript is ZIV, second subscript is SIV, third subscript is MIV

# Classifying Subscripts

- A subscript is **separable** if its indices do not occur in other subscripts
- If two different subscripts contain the same index they are **coupled**
  - $A(i+1, j) = A(k, j) + C$ : Both subscripts are separable
  - $A(i, j, j) = A(i, j, k) + C$ : Second and third subscripts are coupled
- ZIV subscripts are separable, SIV and MIV subscripts may be separable or coupled

- Classifying subscripts allows tests tailored to subscript class
- Coupling indicates complexity in dependence testing
  - Computing dependence information is relatively easy for separable subscripts compared to coupled subscripts

```
    DO I = 1, 100
S1    A(I+1,I+2) = B(I) + C
S2    D(I) = A(I,I) * E
```

# Simple Subscript Tests

```
DO j = 1,100
  A(e1) = A(e2) + B(j)
```

- ZIV test
  - $e1$ and $e2$ are constants or loop invariant symbols
  - If $e1 \neq e2$, then no dependence exists
- SIV test
  - Strong SIV test: $\langle a * i + c_1, a * i + c_2 \rangle$
    - $a$,$c_1$,$c_2$ are constants or loop invariant symbols
    - Example: $\langle 4i + 1, 4i + 5 \rangle$
    - Solution: $d = (c_2 - c_1)/a$ is an integer and $|d| \leq |U_i - L_i|$
  - Weak SIV test: $\langle a_1 * i + c_1, a_2 * i + c_2 \rangle$
    - $a_1$,$a_2$,$c_1$,$c_2$ are constants or loop invariant symbols
    - Example: $\langle 4i + 1, 2i + 5 \rangle$ or $\langle i + 3, 2i \rangle$

# Weak SIV Test

Weak zero SIV: $\langle a_1 * i + c_1, c_2 \rangle$

- Solution: $i = (c_2 - c_1)/a_1$ is an integer and $|i| \leq |U - L|$

```
    DO I = 1, N
S1    Y(I,N) = Y(1,N) + Y(N,N)
```

```
    Y(1,N) = Y(1,N) + Y(N,N)
    DO I = 2, N-1
S1    Y(I,N) = Y(1,N) + Y(N,N)
    Y(N,N) = Y(1,N) + Y(N,N)
```

Weak crossing SIV: $\langle a * i + c_1, -a * i + c_2 \rangle$

- Solution: $i = (c_2 - c_1)/2a$ is an integer and $|i| \leq |U - L|$

```
    DO I = 1, N
S1    A(I) = A(N-I+1) + C
```

```
    DO I = 1, (N+1)/2
S1    A(I) = A(N-I+1) + C
    DO I = (N+1)/2+1, N
S2    A(I) = A(N-I+1) + C
```

# Dependence Testing with GCD

- GCD test is a single LDE MIV subscript test based on elementary number theory
- The Diophantine equation $a_1 i_1 + a_2 i_2 + \cdots + a_n i_n = c$ has an integer solution iff $\gcd(a_1, a_2, \ldots, a_n)$ evenly divides $c$
  - If there is a solution, we can test if it lies within the loop bounds
  - If not, then there is no dependence

```
      for i = 1 to N
S1      a[x*i+k] = ...
S2      ... = a[y*i+m];
```

$x * i_1 + k = y * i_2 + m$,
where $0 \leq i_1, i_2 \leq N$

- If $\text{GCD}(x, y)$ does not divide $(m - k)$, then S1 and S2 **are** independent and can be executed in parallel
- If $\text{GCD}(x, y)$ divides $(m - k)$, then a dependence **may** exist between S1 and S2

Examples:

- $15 * i + 6 * j - 9 * k = 12$, dependence may exist
- $2 * i + 7 * j = 3$, dependence may exist
- $9 * i + 3 * j + 6 * k = 5$ has no integer solution, no dependence

# Problems with GCD Test

- \+ GCD test is a fast elimination test and is often the first step in other dependence tests
- — GCD test provides a necessary but not sufficient condition for data dependence

```
     for i = 1 to 10
S1     a[i] = b[i]+c[i]
S2     d[i] = a[i-20];
```

**Problems**

- — Provides no information on the distance or direction of dependence
- — Ignores loop bounds and GCD is often 1, resulting in false dependences

# Banerjee Test

Computes the minimum and maximum bounds of the LHS, and checks whether the RHS lies within the bounds

- Steps to compute bounds:
  - **Minimum $L$** Multiply a negative (positive) coefficient with the upper (lower) bound of the loop index variable
  - **Maximum $U$** Multiply a negative (positive) coefficient with the lower (upper) bound of the loop index variable
- If $L \leq \text{RHS} \leq U$, **real** solutions exist and dependences may exist, otherwise there is no dependence between the two statements

```
    for (i = 1; i <= 10; i++) {
      for (j = 1; j <= 10; j++) {
S1      g[6*i+2*j] = x[i][j] + 1;
S2      y[i][j] = g[2*i+4*j+2];
      }
    }
```

- Can only guarantee the absence of a dependence
- Indicates existence of real solutions and not integer solutions

# Lamport Test

- Used when there is a single index variable in the subscripts and the coefficients of the index variables are the same
- There is an integer solution only if $d = \frac{c_1 - c_2}{b}$ is an integer
  - Dependence is valid if $|d| \leq U_i - L_i$

```
A[...,b*i+c₁,...] = ...
... = A[...,b*i+c₂,...]
```

```
    for i = 1 to n
      for j = 1 to n
S1      a[i,j] = a[i-1,j+1]
```

```
    for i = 1 to n
      for j = 1 to n
S1      a[i,2j] = a[i-1,2j+1]
```

# Partition-based Dependence Testing

(i) Partition subscripts of a pair of array references into separable and coupled groups
(ii) Classify each subscript as ZIV, SIV, or MIV
(iii) For each separable subscript, apply single subscript test
    - If not done, go to next step
(iv) For each coupled group, apply multiple subscript tests like Delta Test
(v) In any test proves independence, then dependences do not exist
(vi) Otherwise, merge all direction vectors computed in the previous steps into a single set of direction vectors for the two references

G. Goff et al. Practical Dependence Testing, PLDI 1991.

# Delta Test

- Handles multiple, coupled groups of MIV subscripts where the dependences are represented by multiple MIV LDEs
- Notation represents index values at the source and sink

```
DO I = 1, N
  A(I+1) = A(I) + B
```

- Let source iteration be denoted by $I_0$, and sink iteration be denoted by $I_0 + \Delta I$
- Valid dependence implies $I_0 + 1 = I_0 + \Delta I$
- We get $\Delta I = 1 \implies$ Loop-carried dependence with distance vector (1) and direction vector (+)

G. Goff et al. Practical Dependence Testing, PLDI 1991.

# Delta Test

```
DO I = 1, 100
  DO J = 1, 100
    DO K = 1, 100
      A(I+1,J,K) = A(I,J,K+1) + B
```

```
DO I = 1, 100
  DO J = 1, 100
    A(I+1) = A(I) + B(J)
```

Flow dependence

- $I_0 + 1 = I_0 + \Delta I$; $J_0 = J_0 + \Delta J$; $K_0 = K_0 + 1 + \Delta K$
- Solution: $\Delta I = 1$; $\Delta J = 0$; $\Delta K = -1$
- Corresponding direction vector: (+,0,-)

- If a loop index does not appear in a subscript, its distance is unconstrained and its direction is "*" (denotes union of all 3 directions)
- Direction vector assuming flow dependence is (+, *)
  - (*, +) denotes {(+, +), (0, +), (-, +)}
  - (-, +) denotes a level 1 anti-dependence with direction vector (+,-)

# Delta Test

Extract constraints from SIV subscripts and use them for other subscripts

```
DO I = 1, 100
  DO J = 1, 100
    A(I+1, I+J) = A(I, I+J-1) + C
```

```
DO I = 1, N
  DO J = 1, N
    DO K = 1, N
      A(J-I,I+1,J+K) = A(J-I,I,J+K)
```

# Solving Integer Inequalities

- The loop nest inequalities specify a convex polyhedron
  - A polyhedron is convex if for two points in the polyhedron, all points on the line between them are also in the polyhedron
- Data dependence implies a search for integer solutions that satisfy a set of linear inequalities
  - Integer linear programming is an NP-complete problem
- Steps
  1. Use GCD test to check if integer solutions may exist
  2. Use simple heuristics to handle common inequalities
  3. Use a linear integer programming solver that uses a branch-and-bound approach based on Fourier-Motzkin elimination for unsolved inequalities

# Fourier-Motzkin Elimination

**Input** An $n$-dimensional polyhedron $S$ with variables $x_1, x_2, \ldots x_n$

**Goal** Eliminate $x_m$, $m \leq n$

**Output** A polyhedron $S^{'}$ with variables $x_1, x_2, \ldots x_{m-1}, x_{m+1}, \ldots x_n$

**Steps** Let $C$ be all constraints in $S$ involving $x_m$

1. For every pair of a lower bound and upper bound on $x_m \in C$, such as, $L \leq c_1 x_m$ and $c_2 x_m \leq U$, create a new constraint $c_2 L \leq c_1 U$
2. If integers $c_1$ and $c_2$ have a common factor, divide both sides by that factor
3. If the new constraint is not satisfiable, then there is no solution to $S$, i.e., $S$ and $S^{'}$ are empty spaces
4. $S^{'}$ is the set of constraints $S - C$, plus the new constraints generated in Step 2

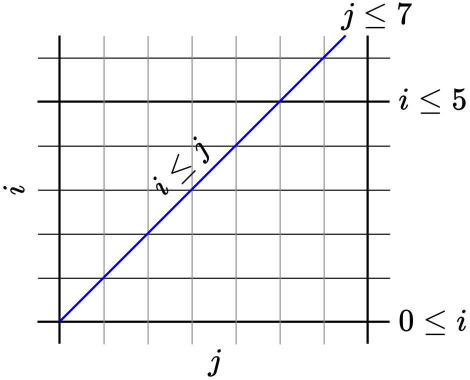
# Example of Fourier-Motzkin Elimination

Consider the code

```
for (i = 0; i <= 5; i++)
  for (j = i; j <= 7; j++)
    Z[j,i] = 0;
```

Goal is to interchange the loops

```
for (j = __; j <= __; j++)
  for (i = __; i <= __; i++)
    Z[j,i] = 0;
```



# Example of Fourier-Motzkin Elimination

```
for (i = 0; i <= 5; i++)
  for (j = i; j <= 7; j++)
    Z[j,i] = 0;
```

Use Fourier-Motzkin elimination to project the 2D space away from the $i$ dimension and onto the $j$ dimension

$$0 \leq i \wedge i \leq 5 \wedge i \leq j \implies 0 \leq j \wedge 0 \leq 5,$$

and we already have $j \leq 7$

The new constraints are:
$0 \leq i, i \leq 5, i \leq j, 0 \leq j, j \leq 7$

Find the loop bounds from the original loop nest: $L_i : 0; U_i : 5, j; L_j : 0; U_j : 7$

```
for (j = 0; j <= 7; j++)
  for (i = 0; i <= min(5,j); i++)
    Z[j,i] = 0;
```

# Use ILP for Dependence Testing

**Algorithm**

Input A convex polyhedron $S$ over variables $v_1, v_2, \ldots v_n$

Output "Yes" if $S$ has an integer solution, "no" otherwise

```
for (i=1; i < 10; i++)
  Z[i] = Z[i+10];
```

Show that there are no two dynamic accesses $i$ and $i'$ with $1 \leq i, i' \leq 9$ and $i = i' + 10$.

# Dependence Testing is Hard

- Most dependence tests assume affine array subscripts
- Unknown loop bounds can lead to false dependences
- Need to be conservative about aliasing
- Triangular loops adds new constraints
- Loop transformations can add additional variables

```
for (i=0; i < N; i++) {
  a[i] = a[i+10];
}
```

How do we compare $N$ and 10?

```
for (i=0; i < N; i++) {
  for (j=0; j < i; j++) {
    a[i][j] = a[j][i];
}
```

Add $j < i$ as a new constraint

```
for (i=L; i < H; i++) {
  a[i] = a[i-1];
}
```

Loop transformations (e.g., normalization) add new variables

# Why is Dependence Analysis Important?

- Dependence information is used to drive important loop transformations
- Goal is to remove dependences or parallelize in the presence of dependences
- We will discuss many transformations (e.g., loop interchange and loop fusion) next

Finding Your Way Around the LLVM Dependence Analysis Zoo: MemorySSA and DependenceAnalysis Tutorial

# References


- R. Allen and K. Kennedy. Optimizing Compilers for Multicore Architectures. Sections 3.1–3.4, Morgan Kaufmann.
- A. Aho et al. Compilers: Principles, Techniques, and Tools. Section 11.6, 2nd edition, Pearson Education.
- Qing Yi. Dependence Testing: Solving System of Diophantine Equations.
- X. Kong et al. The I Test: An Improved Dependence Test or Automatic Parallelization and Vectorization. IEEE Transactions on Parallel and Distributed Systems, 1991.
- M. Wolfe and C.-W. Tseng. The Power Test for Data Dependence. IEEE Transactions on Parallel and Distributed Systems, 1992.
- W. Pugh. A Practical Algorithm for Exact Array Dependence Analysis. Communications of the ACM, 1992.
- Finding Your Way Around the LLVM Dependence Analysis Zoo: MemorySSA and DependenceAnalysis Tutorial