CS 610: Loop Transformations

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Enhancing Program Performance

Possible ideas

- Adequate fine-grained parallelism
  - Multiple pipelined functional units in each core
  - Exploit vector instruction sets (SSE, AVX, AVX-512)
- Adequate parallelism for SMP-type systems
  - Keep multiple asynchronous processors busy with work
- Minimize cost of memory accesses

Role of a Good Compiler

Try and extract performance automatically

Optimize memory access latency

- Code restructuring optimizations
- Prefetching optimizations
- Data layout optimizations
- Code layout optimizations

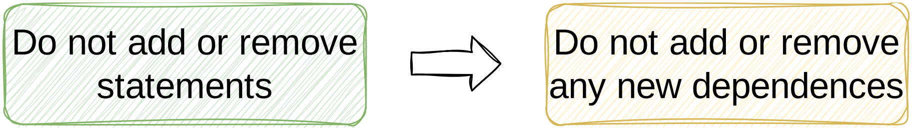
Loop Optimizations

- Loops are one of most **commonly used** constructs in HPC program
- Compiler performs many loop optimization techniques automatically
  - In some cases, source code modifications can enhance optimizer's ability to transform code

Reordering Transformations

A reordering transformation does **not** add or remove statements from a loop nest

- Only reorders the execution of the statements that are already in the loop



A reordering transformation is **valid** if it preserves all existing dependences in the loop

Iteration Reordering and Parallelization

- A transformation that reorders the iterations of a level-k loop, without making any other changes, is valid if the loop carries no dependence
- Each iteration of a loop may be executed in parallel if it carries no dependences

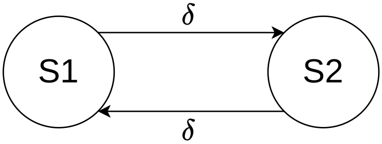
Enhancing Fine-Grained Parallelism

Focus is on vectorization of inner loops

Data Dependence Graph and Parallelization

- If the Data Dependence Graph (DDG) is acyclic, then vectorization of the program is possible and is straightforward
- Otherwise, try to transform the DDG to an acyclic graph

```
      FOR I=1,N
        FOR J=1,M
S1        A(I,J) = B(I-1,J+1) + C
S2        B(I,J) = A(I-1,J-1) + K
```



Loop Interchange (Loop Permutation)

- Switch the nesting order of loops in a perfect loop nest
- Can increase parallelism, can improve spatial locality

```
    DO I = 1, N
      DO J = 1, M
S         A(I,J+1) = A(I,J) + B
```

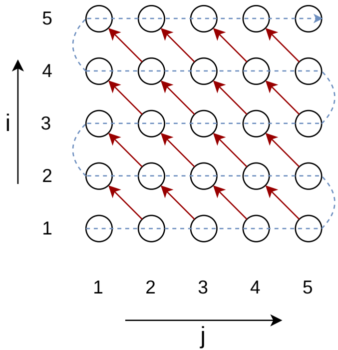
- Dependence is now carried by the outer loop, inner loop can be vectorized

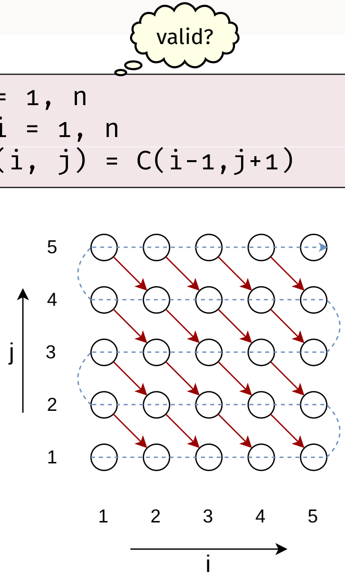
```
    DO J = 1, M
      DO I = 1, N
S         A(I,J+1) = A(I,J) + B
```

Example of Loop Interchange

```
DO i = 1, n
  DO j = 1, n
    C(i, j) = C(i-1,j+1)
```

```
DO j = 1, n
  DO i = 1, n
    C(i, j) = C(i-1,j+1)
```





Validity of Loop Interchange

(i) Construct direction vectors for all possible dependences in the loop to form a direction matrix
   - Identical direction vectors are represented by a single row in the matrix

(ii) Compute direction vectors after permutation

(iii) Permutation of the loops in a perfect nest is legal iff there are no "-" direction as the leftmost non–"0" direction in any direction vector

- Loop interchange is valid for a 2D loop nest if none of the dependence vectors has any negative components
- Interchange is legal: (1,1), (2,1), (0,1), (3,0)
- Interchange is not legal: (1,-1), (3,-2)

```
DO J = 1, M
  DO I = 1, N
    A(I,J+1) = A(I+1,J) + B
```

Validity of Loop Permutation

- Generalization to higher-dimensional loops: Permute all dependence vectors exactly the same way as the intended loop permutation
- If any permuted vector is lexicographically negative, permutation is illegal
- Example: $d_1 = (1, -1, 1)$ and $d_2 = (0, 2, -1)$
  - `ijk` $\rightarrow$ `jik`? $(1, -1, 1) \rightarrow (-1, 1, 1)$: illegal
  - `ijk` $\rightarrow$ `kij`? $(0, 2, -1) \rightarrow (-1, 0, 2)$: illegal
  - `ijk` $\rightarrow$ `ikj`? $(0, 2, -1) \rightarrow (0, -1, 2)$: illegal
  - No valid permutation: `j` cannot be outermost loop (-1 component in d_1), and k cannot be outermost loop (-1 component in d_2)
- A loop nest is **fully** permutable if any permutation transformation to the loop nest is legal

Benefits from Loop Permutation

```
for (i=0; i<n; i++)
  for (j=0; j<n; j++)
    for (k=0; k<n; k++)
      C[i][j] += A[i][k]*B[k][j];
```

Stride	ikj	kij	jik	ijk	jki	kji
C[i][j]	1	1	0	0	n	n
A[i][k]	0	0	1	1	n	n
B[k][j]	1	1	n	n	0	0

Does Loop Interchange/Permutation Always Help?

```
DO i = 1, 10000
  DO j = 1, 1000
    a(i) = a(i) + b(j,i) * c(i)
```

```
DO i = 1, N
  DO j = 1, M
    DO k = 1, L
      a(i+1,j+1,k) = a(i,j,k) + b
```

- Benefits from loop interchange depends on the target machine, the data structures accessed, memory layout and stride patterns
- Optimization choices for the snippet on the right
  - Vectorize J and K
  - Move K outermost and parallelize K with threads
  - Move I innermost and vectorize assuming column-major layout

Loop Shifting

- In a perfect loop nest, if loops at level $i, i+1, \ldots i+n$ carry no dependence, i.e., all dependences are carried by loops at level less than i or greater than $i+n$, then it is always legal to shift these loops inside of loop $i+n+1$
- These loops will not carry any dependences in their new position

Loops i to i+n

+	0	+	0	0	0
0	+	-	+	+	0
0	0	0	0	+	+
0	0	0	0	0	+

Dependence carried by outer loops

Dependence carried by inner loops

Loop Shift for Matrix Multiply

```
    DO I = 1, N
      DO J = 1, N
        DO K = 1, N
S         A(I,J) = A(I,J) + B(I,K)*C(K,J)
```

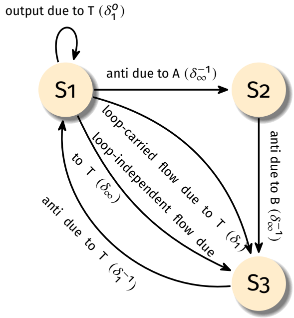
Is the loop nest vectorizable as is?

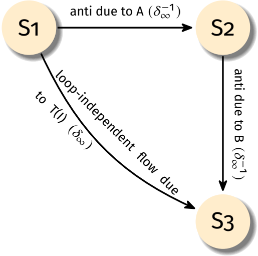
Scalar Expansion

Eliminates dependences that arise from reuse of memory locations at the cost of extra memory

```
    DO I = 1, N
S1    T = A(I)
S2    A(I) = B(I)
S3    B(I) = T
```

```
    DO I = 1, N
S1    $T(I) = A(I)
S2    A(I) = B(I)
S3    B(I) = $T(I)
    T=$T(N)
```





Scalar Expansion

```
DO I = 1, N
  T = T + A(I) + A(I-1)
  A(I) = T
```

```
$T(0) = T
DO I = 1, N
  $T(I) = $T(I-1) + A(I) + A(I-1)
  A(I) = $T(I)
T = $T(N)
```

Can we parallelize the I loop?

Understanding Scalar Expansion

Pros

+ Eliminates dependences due to reuse of memory locations, helps with parallelism

Cons

– Increases memory and addressing overhead

```
DO I = 1, N
  T = A(I) + A(I+1)
  A(I) = T + B(I)
```

can also try forward substitution

⇒

strip mining

```
DO I = 1, N, 64
  DO i = 0, 63
    T = A(I+i) + A(I+1+i)
    A(I+i) = T + B(I+i)
```

⇒

strip loop

```
DO I = 1, N, 64
  DO i = 0, 63
    $T(i) = A(I+i) + A(I+1+i)
    A(I+i) = $T(i) + B(I+i)
```

Strip mining (also known as sectioning) is a special case of 1-D loop tiling

Limits of Scalar Expansion

```
    DO I = 1, 100
S1    T = A(I) + B(I)
S2    C(I) = T + T
S3    T = D(I) - B(I)
S4    A(I+1) = T * T
```

```
    DO I = 1, 100
S1    $T(I) = A(I) + B(I)
S2    C(I) = $T(I) + $T(I)
S3    $T(I) = D(I) - B(I)
S4    A(I+1) = $T(I) * $T(I)
```

Can we vectorize the loop nest?

Scalar Renaming

```
    DO I = 1, 100
S1    T = A(I) + B(I)
S2    C(I) = T + T
S3    T = D(I) - B(I)
S4    A(I+1) = T * T
```

```
    DO I = 1, 100
S1    T1 = A(I) + B(I)
S2    C(I) = T1 + T1
S3    T2 = D(I) - B(I)
S4    A(I+1) = T2 * T2
    T = T2
```

Can we vectorize the loop nest?

Allow Vectorization with Statement Interchange

```
    DO I = 1, 100
S1    T1 = A(I) + B(I)
S2    C(I) = T1 + T1
S3    T2 = D(I) - B(I)
S4    A(I+1) = T2 * T2
    T = T2
```

⇒

```
    DO I = 1, 100
S3    T2 = D(I) - B(I)
S4    A(I+1) = T2 * T2
S1    T1 = A(I) + B(I)
S2    C(I) = T1 + T1
    T = T2
```

⇒

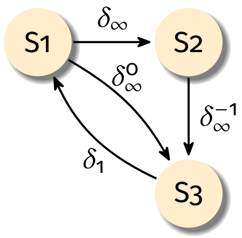
```
S3  T2(1:100) = D(1:100) - B(1:100)
S4  A(2:101) = T2(1:100) * T2(1:100)
S1  T1(1:100) = A(1:100) + B(1:100)
S2  C(1:100) = T1(1:100) + T1(1:100)
    T = T2(100)
```

Array Renaming

```
    DO I = 1, 100
S1    A(I) = A(I-1) + X
S2    Y(I) = A(I) + Z
S3    A(I) = B(I) + C
```

```
    DO I = 1, 100
S1    $A(I) = A(I-1) + X
S2    Y(I) = $A(I) + Z
S3    A(I) = B(I) + C
```

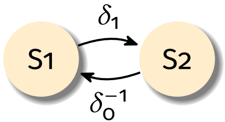
Array renaming requires sophisticated analysis



Node Splitting

```
    DO I = 1, 100
S1    A(I) = X(I+1) + X(I)
S2    X(I+1) = B(I) + 10
```

```
    DO I = 1, 100
S0    $X(I) = X(I+1)
S1    A(I) = $X(I) + X(I)
S2    X(I+1) = B(I) + 10
```



Can we vectorize the loop nest?

Index-Set Splitting

```
DO I = 1, 100
  A(I+20) = A(I) + B
```

strip mining

```
DO I = 1, 100, 20
  DO i = I, I+19
    A(i+20) = A(i) + B
```

An index-set splitting transformation subdivides the loop into different iteration ranges

Loop Peeling

- Splits any problematic iterations (could be first, middle, or last few) from the loop body
- Change from a loop-carried dependence to loop-independent dependence
- Transformed loop carries no dependence, can be parallelized
- Peeled iterations execute in the original order, transformation is always legal to perform

```
DO I = 1, N
  A(I) = A(I) + A(1)
```

```
A(1) = A(1) + A(1)
DO I = 2, N
  A(I) = A(I) + A(1)
```

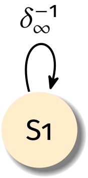
Loop splitting

Loop Splitting

```
DO I = 1, N
  A(I) = A(N/2) + B(I)
```

assume N is divisible by 2

```
M = N/2
DO I = 1, M-1
  A(I) = A(N/2) + B(I)
A(M) = A(N/2) + B(I)
DO I = M+1, N
  A(I) = A(N/2) + B(I)
```



Section-Based Splitting

```
    DO I = 1,N
      DO J = 1, N/2
S1      B(J,I) = A(J,I) + C
      DO J = 1,N
S2      A(J,I+1) = B(J,I) + D
```

⇒

```
    DO I = 1,N
      DO J = 1, N/2
S1      B(J,I) = A(J,I) + C
      DO J = 1,N/2
S2      A(J,I+1) = B(J,I) + D
      DO J = N/2+1, N
S3      A(J,I+1) = B(J,I) + D
```

S3 is independent

⇒

```
    DO I = 1,N
      DO J = N/2+1, N
S3      A(J,I+1) = B(J,I) + D
    DO I = 1,N
      DO J = 1,N/2
S1      B(J,I) = A(J,I) + C
      DO J = 1, N/2
S2      A(J,I+1) = B(J,I) + D
```

⇒

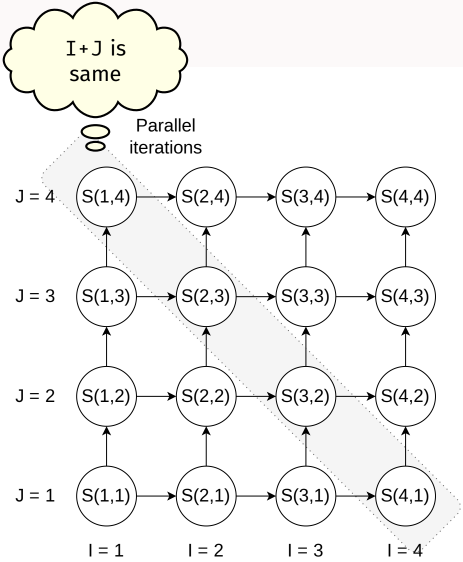
```
    M = N/2
S3  A(M+1:N,2:N+1) = B(M+1:N,1:N) + D
    DO I = 1, N
S1    B(1:M,I) = A(1:M,I) + C
S2    A(1:M,I+1) = B(1:M,I) + D
```

cannot vectorize I

Loop Skewing

```
    DO I = 1, N
      DO J = 1, N
S       A(I,J) = A(I-1,J) + A(I,J-1)
```

Which loops carry dependences?

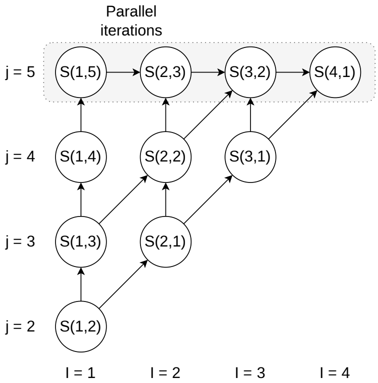


Loop Skewing

j=I+J

```
    DO I = 1, N
      DO j = I+1, I+N
S       A(I,j-I) = A(I-1,j-I) + A(I,j-I-1)
```

What are the dependences now? Which loop carries the dependence?



Loop skewing skews the inner loop relative to the outer loop by adding the index of the outer loop times a skewing factor f to the bounds of the inner loop and subtracting the same value from all the uses of the inner loop index

Perform Loop Interchange

Given a dependency vector (a, b), skewing transforms it to $(a, fa + b)$

```
    DO I = 1, N
      DO j = I+1, I+N
S       A(I,j-I) = A(I-1,j-I) + A(I,j-I-1)
```

$$\Downarrow$$

can use Fourier-Motzkin elimination

```
    DO j = 2, N+N
      DO I = max(1,j-N), min(N,j-1)
S       A(I,j-I) = A(I-1,j-I) + A(I,j-I-1)
```

Understanding Loop Skewing

Pros

+ Reshapes the iteration space to find possible parallelism
+ Preserves lexicographic order of the dependences, is always legal
+ Allows for loop interchange in future

Cons

– Resulting iteration space can be trapezoidal
– Irregular loops are not very amenable for vectorization
– Need to be careful about load imbalance

Loop Unrolling (Loop Unwinding)

- Reduce number of iterations of loops
- Add statement(s) to do work of missing iterations
- JIT compilers try to perform unrolling at run-time

```
for (i=0; i<n; i++) {
  for (j=0; j<n; j++) {
    y[i] = y[i] + a[i][j]*x[j];
  }
}
```

```
for (i=0; i<n; i++) {
  for (j=0; j<n; j+=4) {
    y[i] = y[i] + a[i][j]*x[j]
           + a[i][j+1]*x[j+1]
           + a[i][j+2]*x[j+2]
           + a[i][j+3]*x[j+3];
  }
}
```

4-way inner loop unrolling

Outer Loop Unrolling + Inner Loop Jamming

```
for (i=0; i<2*n; i++) {
  for (j=0; j<m; j++) {
    loop-body(i,j);
  }
}
```

```
for (i=0; i<2*n; i+=2) {
  for (j=0; j<m; j++) {
    loop-body(i,j);
  }
  for (j=0; j<m; j++) {
    loop-body(i+1,j);
  }
}
```

```
for (i=0; i<2*n; i+=2) {
  for (j=0; j<m; j++) {
    loop-body(i,j);
    loop-body(i+1,j);
  }
}
```

2-way outer unroll does not increase operation-level parallelism in the inner loop

Is Loop Unroll and Jam Legal?

```
DO I = 1, N
  DO J = 1, M
    A(I,J) = A(I-1,J+1)+C
```

```
DO I = 1, N, 2
  DO J = 1, M
    A(I,J) = A(I-1,J+1)+C
    A(I+1,J) = A(I,J+1)+C
```

Validity Condition for Loop Unroll and Jam

- Complete unroll and jam of a loop is equivalent to a loop permutation that moves that loop innermost, without changing order of other loops
- If such a loop permutation is valid, unroll and jam of the loop is valid
- Example: 4D loop `ijkl`; $d_1 = (1, -1, 0, 2)$, $d_2 = (1, 1, -2, -1)$
  - i $d_1 \rightarrow (-1, 0, 2, 1), \implies$ invalid to unroll and jam
  - j $d_1 \rightarrow (1, 0, 2, -1); d_2 \rightarrow (1, -2, -1, 1), \implies$ valid to unroll and jam
  - k $d_1 \rightarrow (1, -1, 2, 0); d_2 \rightarrow (1, 1, -1, -2), \implies$ valid to unroll and jam
  - l d_1 and d_2 are unchanged; innermost loop can always be unrolled

Understanding Loop Unrolling

Pros

- + Small loop bodies are problematic, reduces control overhead of loops
- + Increases operation-level parallelism in loop body
- + Allows other optimizations like reuse of temporaries across iterations

Cons

- – Increases the executable size
- – Increases register usage
- – May prevent function inlining

Loop Tiling (Loop Blocking)

- Improve data reuse by chunking the data in to smaller tiles (blocks)
  - All the required blocks are supposed to fit in the cache
- Performs strip mining in multiple array dimensions
- Tries to exploit spatial and temporal locality of data
- Determining the tile size
  - Requires accurate estimate of array accesses and the cache size of the target machine
  - Loop nest order also influences performance
  - Difficult theoretical problem, usually heuristics are applied
  - Cache-oblivious algorithms make efficient use of cache without explicit blocking

```
for (i = 0; i < N; i++) {
  ...
}
```

```
for (ii = 0; ii < N; ii+=B) {
  for (i = ii; i < min(N,ii+B), i++) {
    ...
  }
}
```

Validity Condition for Loop Tiling

- A band of loops is fully permutable if all permutations of the loops in that band are legal
- A contiguous band of loops can be tiled if they are fully permutable

```
for (i = 0; i < n; i++)
  for (j = 0; j < n; j++)
    for (k = 0; k < n; k++)
      loop_body(i,j,k)
```

- Example: $d = (1, 2, -3)$
  - Tiling all three loops `ijk` is not valid, since the permutation `kij` is invalid
  - 2D tiling of band `ij` is valid
  - 2D tiling of band `jk` is valid

```
for (it = 0; it < n; it+=T)
  for (jt = 0; jt < n; jt+=T)
    for (i = it; i < it+T; i++)
      for (j = jt; j < jt+T; j++)
        for (k = 0; k < n; k++)
          loop_body(i,j,k)
```

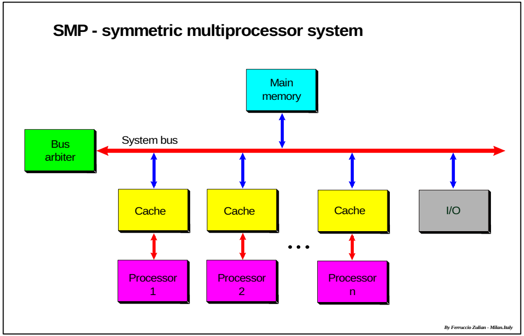
Enhancing Coarse-Grained Parallelism

Focus is on parallelization of outer loops

Find Work for Threads

Setup
- Symmetric multiprocessors with shared memory
- Threads are running on each core and are coordinating execution with occasional synchronization

Challenge Balance the granularity of parallelism with communication overheads



Challenges in Coarse-Grained Parallelism

Minimize communication and synchronization overhead while evenly load balancing across the processors

Running everything on one processor achieves minimal communication and synchronization overhead

Very fine-grained parallelism achieves good load balance, but benefits may be outweighed by frequent communication and synchronization

Challenges in Coarse-Grained Parallelism

Minimize communication and synchronization overhead while evenly load balancing across th[...] ces[...]

Running eve[...] achieves mi[...] synchroniza[...]

[...]chieves good [...] be [...]unication

An optimizing compiler is expected to find the sweet spot

Privatization

- Privatization is similar to scalar expansion
- Temporaries can be made local to each iteration

```
    DO I = 1,N
S1    T = A(I)
S2    A(I) = B(I)
S3    B(I) = T
```

```
    PARALLEL DO I = 1,N
      PRIVATE t
S1    t = A(I)
S2    A(I) = B(I)
S3    B(I) = t
```

Privatization

A scalar variable x in a loop L is privatizable if every path from the entry of L to a use of x in the loop passes through a definition of x

- No use of the variable is upward exposed, i.e., the use never reads a value that was assigned outside the loop
- No use of the variable is from an assignment in an earlier iteration

Computing upward-exposed variables from a block BB

$$up(BB) = use(BB) \cup \left(\neg def(BB) \cap \bigcup_{y \in succ(BB)} up(y)\right)$$

Computing privatizable variables for a loop body B where BB_0 is the entry block

$$private(B) = \neg up(BB_0) \cap \left(\bigcup_{y \in B} def(y)\right)$$

Privatization

- If all dependences carried by a loop involve a privatizable variable, then loop can be parallelized by making the variables private
- Preferred compared to scalar expansion
  - Less memory requirement
  - Scalar expansion may suffer from false sharing
- However, there can be situations where scalar expansion works but privatization does not

Comparing Privatization and Scalar Expansion

```
DO I = 1, N
  T = A(I) + B(I)
  A(I-1) = T
```

Scalar expansion $\Longrightarrow$

```
PARALLEL DO I = 1, N
  T$(I) = A(I) + B(I)
  A(I-1) = T$(I)
```

$\Downarrow$ Privatization

```
DO I = 1, N
  PRIVATE T
  T = A(I) + B(I)
  A(I-1) = T
```

$\Downarrow$

```
PARALLEL DO I = 1, N
  T$(I) = A(I) + B(I)
PARALLEL DO I = 1, N
  A(I-1) = T$(I)
```

Loop Distribution (Loop Fission)

```
    DO I = 1, 100
      DO J = 1, 100
S1        A(I,J) = B(I,J) + C(I,J)
S2        D(I,J) = A(I,J-1) * 2.0
```

```
    DO I = 1, 100
      DO J = 1, 100
S1        A(I,J) = B(I,J) + C(I,J)

      DO J = 1, 100
S2        D(I,J) = A(I,J-1) * 2.0
```

Eliminates loop-carried dependences

Validity Condition for Loop Distribution

A loop with two statements can be distributed if there are no dependences from any instance of the **later** statement to any instance of the **earlier** one

- Sufficient (but not necessary) condition
- Generalizes to more statements

```
    DO I = 1, N
S1    A(I) = B(I) + C(I)
S2    E(I) = A(I+1) * D(I)
```

```
    DO I = 1, N
S1    A(I) = B(I) + C(I)
S2    E(I) = A(I-1) * D(I)
```

Performing Loop Distribution

Steps

(i) Build the DDG

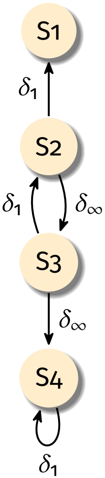
(ii) Identify strongly-connected components (SCCs) in the DDG

(iii) Make each SCC a separate loop

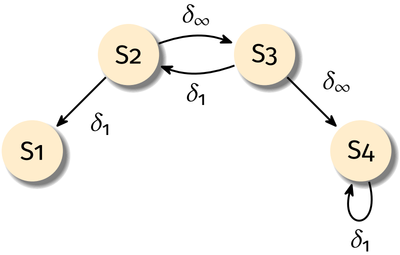
(iv) Arrange the new loops in a topological order of the DDG

```
    DO I = 1, N
S1    A[I] = A[I] + B[I-1]
S2    B[I] = C[I-1] + X
S3    C[I] = B[I] + Y
S4    D[I] = C[I] + D[I-1]
```

⟹



⟹



⇓

```
    DO I = 1, N
S2    B[I] = C[I-1] + X
S3    C[I] = B[I] + Y
    DO I = 1, N
S1    A[I] = A[I] + B[I-1]
    DO I = 1, N
S4    D[I] = C[I] + D[I-1]
```

Understanding Loop Distribution

Pros

- + Execute source of a dependence before the sink
- + Reduces the memory footprint of the original loop for both data and code
- + Improves opportunities for vectorization

Cons

- – Can increase the synchronization required between dependence points

Loop Alignment

Unlike loop distribution, realign the loop to compute and use the values in the same iteration

```
    DO I = 2, N
S1    A(I) = B(I) + C(I)
S2    D(I) = A(I-1) * 2.0
```

cannot be parallelized

```
    DO i = 1, N+1
      if i > 1 && i < N+1
S1      A(i) = B(i) + C(i)
      if i < N
S2      D(i+1) = A(i) * 2.0
```

carried dependence becomes loop independent

Can Loop Alignment Eliminate All Carried Dependences?

```
    DO I = 1, N
S1    A(I) = B(I) + C
S2    B(I+1) = A(I) + D
```

A is aligned, B is misaligned

⟹

```
    DO i = 1, N+1
      if i > 1
S2      B(i) = A(i-1) + D
      if i < N+1
S1      A(i) = B(i) + C
```

```
    DO I = 1, N
S1    A(I+1) = B(I) + C
S2    X(I) = A(I+1) + A(I)
```

⟹

```
    DO i = 0, N
      if i > 0
S1      A(i+1) = B(i) + C
      if i < N
S2      X(i+1) = A(i+2) + A(i+1)
```

Loop Fusion (Loop Jamming)

```
    DO I = 1, N
S1    A(I) = B(I) + 1
S2    C(I) = A(I) + C(I-1)
S3    D(I) = A(I) + X
```

⟹

```
L1  DO I = 1, N
      A(I) = B(I) + 1
L2  DO I = 1, N
      C(I) = A(I) + C(I-1)
L3  DO I = 1, N
      D(I) = A(I) + X
```

⟹

```
L13  DO I = 1, N
       A(I) = B(I) + 1
       D(I) = A(I) + X
L2   DO I = 1, N
       C(I) = A(I) + C(I-1)
```

Validity Condition for Loop Fusion

- Consider a loop-independent dependence between statements in two different loops (i.e., from S1 to S2)
- A dependence is fusion-preventing if fusing the two loops causes the dependence to be carried by the combined loop in the reverse direction (from S2 to S1)

```
    DO I = 1, N
S1    A(I) = B(I) + C
    DO I = 1, N
S2    D(I) = A(I+1) + E
```

loop-independent flow dependence

```
    DO I = 1, N
S1    A(I) = B(I) + C
S2    D(I) = A(I+1) + E
```

backward loop-carried anti dependence

Understanding Loop Fusion

Pros

+ Reduce overhead of loops
+ May improve temporal locality

```
    DO I = 1, N
S1    A(I) = B(I) + C
    DO I = 1, N
S2    D(I) = A(I-1) + E
```

Cons

– May decrease data locality in the fused loop

```
    DO I = 1, N
S1    A(I) = B(I) + C
S2    D(I) = A(I-1) + E
```

Loop Interchange

```
DO I = 1, N
  DO J = 1, M
    A(I+1,J) = A(I,J) + B(I,J)
```

Parallelizing J is good for vectorization, but not for coarse-grained parallelism

⟹

```
DO J = 1, M
  DO I = 1, N
    A(I+1,J) = A(I,J) + B(I,J)
```

Dependence-free loops should move to the outermost level

⇓

```
PARALLEL DO J = 1, M
  DO I = 1, N
    A(I+1,J) = A(I,J) + B(I,J)
```

Condition for Loop Interchange

In a perfect loop nest, a loop can be parallelized at the outermost level if and only if the column of the direction matrix for that nest contains only "0" entries

```
DO I = 1, N
  DO J = 1, M
    A(I+1,J+1) = A(I,J) + B(I,J)
```

Code Generation Strategy

(i) Continue till there are no more columns to move
  - Choose a loop from the direction matrix that has all “0” entries in the column
  - Move it to the outermost position
  - Eliminate the column from the direction matrix

(ii) Pick loop with most “+” entries, move to the next outermost position
  - Generate a sequential loop
  - Eliminate the column
  - Eliminate any rows that represent dependences carried by this loop

(iii) Repeat from Step (i)

Code Generation Example

```
DO I = 1, N
  DO J = 1, M
    DO K = 1, L
      A(I+1,J,K) = A(I,J,K) + X1
      B(I,J,K+1) = B(I,J,K) + X2
      C(I+1,J+1,K+1) = C(I,J,K) + X3
```

```
DO I = 1, N
  PARALLEL DO J = 1, M
    DO K = 1, L
      A(I+1,J,K) = A(I,J,K) + X1
      B(I,J,K+1) = B(I,J,K) + X2
      C(I+1,J+1,K+1) = C(I,J,K) + X3
```

How did we pick loop J for parallelization?

How can we parallelize this loop?

```
DO I = 2, N+1
  DO J = 2, M+1
    DO K = 1, L
      A(I,J,K) = A(I,J-1,K+1) + A(I-1,J,K+1)
```

No single loop carries all the dependences, so we can only parallelize loop K

Loop Reversal

```
DO I = 2, N+1
  DO J = 2, M+1
    DO K = 1, L
      A(I,J,K) = A(I,J-1,K+1) + A(I-1,J,K+1)
```

$$\Downarrow$$

```
DO I = 2, N+1
  DO J = 2, M+1
    DO K = L, 1, -1
      A(I,J,K) = A(I,J-1,K+1) + A(I-1,J,K+1)
```

- When the iteration space of a loop is reversed, the direction of dependences within that reversed iteration space are also reversed
  - A "+" dependence becomes a "-" dependence, and vice versa
- We cannot perform loop reversal if the loop carries a dependence

Perform Interchange after Loop Reversal

```
DO I = 2, N+1
  DO J = 2, M+1
    DO K = L, 1, -1
      A(I,J,K) = A(I,J-1,K+1) + A(I-1,J,K+1)
```

$\Downarrow$

```
DO K = L, 1, -1
  DO I = 2, N+1
    DO J = 2, M+1
      A(I,J,K) = A(I,J-1,K+1) + A(I-1,J,K+1)
```

increases options for performing other optimizations

Which Transformations are Most Important?

- Selecting the best loops for parallelization is a NP-complete problem
- Flow dependences are difficult to remove
  - Try to reorder statements as in loop peeling, loop distribution
- Techniques like scalar expansion, privatization can be useful
  - Loops often use scalars for temporary values

+	+	O	O
+	O	+	O
+	O	O	+
O	+	O	O
O	O	+	O
O	O	O	+

carries the most dependences

Unimodular Transformations

Challenges in Applying Transformations

- We have discussed transformations (legality and benefits) in isolation
- Compilers need to apply compound transformations (e.g., loop interchange followed by reversal)
- It is challenging to decide on the desired transformations and their order of application
  - Choice and order is sensitive to the program input, a priori order does not work

Unimodular Transformations

- A unimodular matrix is a square integer matrix having determinant 1 or -1 (e.g., $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$)
- Few loop transformations can be modeled as matrix transformations involving unimodular matrices
  - Loop interchange maps iteration (i, j) to iteration (j, i)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} j \\ i \end{bmatrix}$$

  - Given transformation T is linear, the transformed dependence is given by Td where d is the dependence vector in the original iteration space

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}$$

  - The transformation matrix for loop reversal of the outer loop i in a 2D loop nest is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
  - The transformation matrix for loop skewing of a 2D loop nest (i, j) is the identity matrix T with $T_{j,i}$ equal to f, where we skew loop j with respect to loop i by a factor f

M. Wolf and M. Lam. A Loop Transformation Theory and an Algorithm to Maximize Parallelism. TPDS'91.

Example of Loop Skewing

Original

```
FOR I=1,5
  FOR J=1,5
    A(I,J) = A(I-1,J) + A(I,J-1)
```

Dependences $D = \{(1,0),(0,1)\}$

Skewed

```
FOR I=1,5
  FOR j=I+1,I+5
    A(I,j-I) = A(I-1,j-I) + A(I,j-I-1)
```

Transformation matrix = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Dependences $D' = TD = \{(1,1),(0,1)\}$

Representing Compound Transformations

```
DO I = 1, N
  DO J = 1, N
    A(I,J) = A(I-1,J+1) + C
```

Loop interchange is illegal because

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let us try loop interchange followed by loop reversal. The transformation matrix T is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Applying T to the loop nest is legal because

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
DO J = N, 1, -1
  DO I = 1, N
    A[I,J] = A[I-1,J+1] + C
```

Challenges for Real-World Compilers

- Conditional execution
- Symbolic loop bounds
- Indirect memory accesses
- ...

References

R. Allen and K. Kennedy. Optimizing Compilers for Multicore Architectures. Chapters 5–6, Morgan Kaufmann.

S. Midkiff. Automatic Parallelization: An Overview of Fundamental Compiler Techniques. Springer Cham.