CS 771A: Intro to Machine Learning, IIT Kanpur				**Quiz II** (29 Mar 2023)
Name	MELBO			**20 marks**
Roll No	230001	Dept.	AWSM	Page **1** of **2**

Instructions:
1. This question paper contains 1 page (2 sides of paper). Please verify.
2. Write your name, roll number, department above in **block letters neatly with ink**.
3. Write your final answers neatly **with a blue/black pen**. Pencil marks may get smudged.
4. Don't overwrite/scratch answers especially in MCQ – such cases will get straight 0 marks.
5. Do not rush to fill in answers. You have enough time to solve this quiz.

Q1. The random variable X is sampled from the uniform distribution over the interval $[0,1]$ i.e., the probability density function of X looks like $p(x) = 1$ if $x \in [0,1]$ and $p(x) = 0$ if $x < 0$ or $x > 1$. The r.v. Y is such that $\mathbb{E}[X^2 + Y^2] = 1$ and $\text{Var}[Y] = \frac{5}{9}$. Y need not be independent of X and may have different support than X. Calculate the following (no derivation needed). **(4 x 1 = 4 marks)**

$\mathbb{E}[X] = 1/2$ $\quad$ $\text{Var}[X] = 1/12$

$\mathbb{E}[Y] = 1/3 \text{ or } -1/3$ $\quad$ $\mathbb{E}[X+Y] = 5/6 \text{ or } 1/6$

Q2. Melbo is playing a cricket match with his rival Oblem and they use a coin to decide who bats first. The coin has $\mathbb{P}[\text{Toss} = \text{H}] = p$ and $\mathbb{P}[\text{Toss} = \text{T}] = 1 - p$ but Melbo is suspicious that Oblem is cheating and that $p \neq \frac{1}{2}$. To promote fair play, Melbo proposes an unconventional process. The coin is tossed twice (independently), and the outcome is decided as follows:

1. If the tosses are HT (in that order), then Melbo bats first (let's call this event M).
2. If the tosses are TH (in that order), then Oblem bats first (let's call this event O).
3. If the tosses are TT or HH, then a Null result is declared (let's call this event N).

The coin is tossed twice. Calculate the following. ($\neg N$ is the event "not N") **(4 x 1 = 4 marks)**

$\mathbb{P}[N] = 2p^2 - 2p + 1$ $\quad$ $\mathbb{P}[M \mid \neg N] = 1/2$

Recall that p is unknown and it is suspected that $p \neq 0.5$
so $\mathbb{P}[N]$ must be in terms of p and we cannot claim $\mathbb{P}[N] = 0.5$

$\mathbb{P}[O \mid \neg N] = 1/2$ $\quad$ $\mathbb{P}[M \mid N] = 0$

Q3. Oblem still wants to cheat and proposes another way to decide batting. The Poisson distribution allows us to sample non-negative integers i.e., $0,1,2,3,4 \ldots$. The distribution has a single parameter $\lambda > 0$ and if X is a $\text{Poisson}(\lambda)$ random variable, then for any $n \in \{0,1,2,3,4 \ldots\}$, we have

$$\mathbb{P}[X = n] = \frac{\lambda^n \exp(-\lambda)}{n!}$$

Oblem chooses $\lambda_1, \ldots, \lambda_D > 0$. Then, D variables $X_i \sim \text{Poisson}(\lambda_i)$ are independently drawn. If the sum of the drawn variables is equal to zero i.e., $\sum_{d \in [D]} X_d = 0$, then Melbo will bat first (call it event M) else Oblem will bat first (call it event O). Find expressions for $\mathbb{P}[O]$ and $\mathbb{P}[M]$ in terms of $\lambda_1, \ldots, \lambda_D$. Give brief derivation. Melbo suspects that Oblem may cheat by choosing values of $\lambda_1, \ldots, \lambda_D$ that maximize Oblem's chances of batting first. To avoid this, Melbo imposes a constraint that the λ_d values must satisfy $\sum_{d\in[D]} \lambda_d^2 \leq 1$. Set up an optimization problem to find the values of $\lambda_1, \ldots, \lambda_D > 0$ that maximize Oblem's chances of batting first, subject to Melbo's constraint. Find the value of that probability too. Give brief derivations. Note that $0! \stackrel{\text{def}}{=} 1, \lambda_d^0 \stackrel{\text{def}}{=} 1$. **(12 marks)**

Expressions for win probability in terms of $\lambda_1, \ldots, \lambda_D$ (1 + 1 = 2 marks)

$$\mathbb{P}[O] = 1 - \exp\left(-\sum_{d\in[D]} \lambda_d\right) \quad \mathbb{P}[M] = \exp\left(-\sum_{d\in[D]} \lambda_d\right)$$

Brief derivation for win probability in terms of $\lambda_1, \ldots, \lambda_D$ (2 marks)

Melbo bats first if $\sum_{d\in[D]} X_d = 0$ which can only happen if each $X_d = 0$ since all $X_d \geq 0$. We have $\mathbb{P}[X_d = 0] = \frac{\lambda_d^0 \exp(-\lambda_d)}{0!} = \exp(-\lambda_d)$. Since the draws are independent, we get

$$\mathbb{P}[M] = \mathbb{P}\left[\sum_{d\in[D]} X_d = 0\right] = \prod_{d\in[D]} \mathbb{P}[X_d = 0] = \exp\left(-\sum_{d\in[D]} \lambda_d\right)$$

Highest win probability for Oblem subject to Melbo's constraint (2 marks)

$$\max_{\substack{\lambda_i > 0 \\ \sum_{d\in[D]} \lambda_d^2 \leq 1}} \mathbb{P}[O] = 1 - \exp\left(-\sqrt{D}\right)$$

The value of $\lambda_1, \ldots, \lambda_D$ that satisfy Melbo's constraint at which Oblem's win probability is maximized (2 marks)

$$[\lambda_1, \ldots \lambda_D] = \left[1/\sqrt{D}, 1/\sqrt{D}, \ldots, 1/\sqrt{D}\right]$$

Brief derivation for Oblem's highest win probability subject to Melbo's constraint (4 marks)

Maximizing $\mathbb{P}[M]$ is equivalent to minimizing $-\sum_{d\in[D]} \lambda_d$. However, we also have constraints $\sum_{d\in[D]} \lambda_d^2 \leq 1$ and $\lambda_d \geq 0$ for all $d \in [D]$. Introducing Lagrange multipliers $\alpha \geq 0$ and $\beta_d \geq 0$ gives us the Lagrangian $\mathcal{L}(\boldsymbol{\lambda}, \alpha, \boldsymbol{\beta}) = -\boldsymbol{\lambda}^\top \mathbf{1} + \alpha(\|\boldsymbol{\lambda}\|_2^2 - 1) - \boldsymbol{\beta}^\top \boldsymbol{\lambda}$ where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_D]$, $\mathbf{1} = [1,1, \ldots, 1] \in \mathbb{R}^D$ and $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_D]$.

Stationarity tells us that $\partial\mathcal{L}/\partial\boldsymbol{\lambda} = \mathbf{0} = -1 + 2\alpha \cdot \boldsymbol{\lambda} - \boldsymbol{\beta}$ i.e., $\boldsymbol{\lambda} = (\boldsymbol{\beta} + \mathbf{1})/2\alpha$. Substituting this back into the Lagrangian eliminates $\boldsymbol{\lambda}$ and gives us the following dual problem.

$$\max_{\alpha, \boldsymbol{\beta} \geq 0} -\frac{(\boldsymbol{\beta}^\top \mathbf{1} + D)}{2\alpha} + \left(\frac{\|\boldsymbol{\beta} + \mathbf{1}\|_2^2}{4\alpha} - \alpha\right) - \frac{\|\boldsymbol{\beta}\|_2^2 + \boldsymbol{\beta}^\top \mathbf{1}}{2\alpha} = \min_{\alpha, \boldsymbol{\beta} \geq 0} \frac{\|\boldsymbol{\beta} + \mathbf{1}\|_2^2}{4\alpha} + \alpha$$

For any fixed value of $\alpha \geq 0$, applying the QUIN rule tells us that the optimal value of $\beta_d = 0$. Thus, we are left to solve $\min_{\alpha \geq 0} \frac{D}{4\alpha} + \alpha$ which is optimized at $\alpha = \sqrt{D/4}$ which gives $\lambda_d = 1/\sqrt{D}$.