CS 771A: Intro to Machine Learning, IIT Kanpur				**Quiz I** (1 Feb 2022)
Name	MELBO			**20 marks** Page **1** of **2**
Roll No	220001	Dept.	AWSM	

Instructions:
1. This question paper contains 1 page (2 sides of paper). Please verify.
2. Write your name, roll number, department above in **block letters neatly with ink**.
3. Write your final answers neatly **with a blue/black pen**. Pencil marks may get smudged.
4. Don't overwrite/scratch answers especially in MCQ – such cases will get straight 0 marks.
5. Do not rush to fill in answers. You have enough time to solve this quiz.

Q1. Write **T** or **F** for True/False (write **only in the box on the right-hand side**) **(6 x 1 = 6 marks)**

1	If $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^2$ are two non-empty convex sets, then their union i.e., $\mathcal{P} \cup \mathcal{Q}$ can never be convex.	F
2	$f: \mathbb{R} \to \mathbb{R}$ is convex, non-differentiable and $-1 \in \partial f(x_0)$ (i.e., -1 is a subgradient), then f must be decreasing around x_0 i.e., for some $\epsilon > 0$, $f(x_0 - \epsilon) > f(x_0 + \epsilon)$.	F
3	If $\mathcal{C} \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{R}^2 : 2 \leq x + y \leq 3\}$ is the set of all 2D vectors whose sum of coordinates is between 2 and 3, then $\mathcal{C}$ is a convex set.	T
4	If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are two vectors such that $\|\mathbf{u}\|_2 \leq 1$ and $\|\mathbf{v}\|_2 < 2$ then it is possible that $\mathbf{u}^\top \mathbf{v} > 2.5$ for certain values of $\mathbf{u}, \mathbf{v}$.	F
5	If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are two vectors such that $\|\mathbf{u}\|_2 \leq 1$ and $\|\mathbf{v}\|_2 < 2$ then it is possible that $\|\mathbf{u} - \mathbf{v}\|_2 > 2.5$ for certain values of $\mathbf{u}, \mathbf{v}$.	T
6	Classification models such as SVMs are more confident about their predictions on data points that lie very close to their decision boundary.	F

Q2. (Sliding parabolas) Consider the two functions $f(x) = (x - a)^2 + b$ and $g(x) = x^4 + x^2$.

Find values of $a, b \in \mathbb{R}$ such that the functions f, g share a tangent at $x = 1$.	$a = -2$	$b = -7$
For the values you find above, find out the value of the function f at $x = 1$.	$f(1) = 2$	
Write down the equation of the (shared) tangent of the functions at $x = 1$.	$6x - 4$	
Find values of $a, b \in \mathbb{R}$ such that the functions f, g share a stationary point x_0 such that $f(x_0) = g(x_0)$.	$a = 0$	$b = 0$

Note that your equation for the tangent must not contain variables like a, b and should contain only absolute constants. Write your answers only in the space provided. **(2 + 1 + 1 + 2 = 6 marks)**

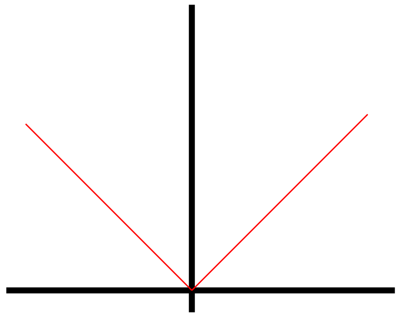
Q3 (Robust regression) Training with the absolute loss can make regression models less sensitive to outliers. Let us analyse a simplified version of the objective function used in this technique.

a. Consider the function $f(w) = |b \cdot w - a|$ where $a, b, w \in \mathbb{R}$. For any value of $a, b, w^0 \in \mathbb{R}$, write down an expression for the (entire) subdifferential of f at w^0. No derivation needed.

b. Consider the function $g(\mathbf{w}) = \frac{1}{2}\|\mathbf{w}\|_2^2 + |\mathbf{w}^\top \mathbf{x} - y|$ where $\mathbf{w}, \mathbf{x} \in \mathbb{R}^d$ and $y \in \mathbb{R}$. Write down an expression for the (entire) subdifferential of g at $\mathbf{w}^0$ for any value of $\mathbf{w}^0, \mathbf{x} \in \mathbb{R}^d$ and $y \in \mathbb{R}$. No derivation needed. Write your answer in the space given. **(2 + 2 = 4 marks)**

The function $a(x) \overset{\text{def}}{=} |x|$ is convex with a single point of non-differentiability at $x = 0$. Thus, we have $\partial a(x) = \begin{cases} \{-1\} & x < 0 \\ [-1,1] & x = 0 \\ \{1\} & x > 0 \end{cases}$. Applying the chain rule tells us that:

$$\partial f(w^0) = \begin{cases} \{-b\} & bw^0 < a \\ S & bw^0 = a \\ \{b\} & bw^0 > a \end{cases}$$



where $S = \{b \cdot g : g \in [-1,1]\}$.

Note that the subdifferential is a set so ideally, we should write $\{-b\}$ and not just $-b$.

The first term in the function $\frac{1}{2}\|\mathbf{w}\|_2^2$ is differentiable. Applying the chain rule and the sum rule tells us that:

$$\partial g(\mathbf{w}^0) = \begin{cases} \{\mathbf{w}^0 - \mathbf{x}\} & \mathbf{w}^\top \mathbf{x} < y \\ S & \mathbf{w}^\top \mathbf{x} = y \\ \{\mathbf{w}^0 + \mathbf{x}\} & \mathbf{w}^\top \mathbf{x} > y \end{cases}$$

where $S = \{\mathbf{w}^0 + g \cdot \mathbf{x} : g \in [-1,1]\}$

Q4. (Vector line-up) Give examples of 4D vectors (fill-in the 4 boxes) with the following properties. Any example will get full marks so long as it satisfies the property mentioned in the question part. Your answers to the parts a, b, c, d may be same/different. **(4 x 1 = 4 marks)**

a. A vector $\mathbf{v} \in \mathbb{R}^4$ such that $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v}^\top \mathbf{u} = 0$ where $\mathbf{u} = (1,-1,1,-1) \in \mathbb{R}^4$.

1	1	1	1

b. A vector $\mathbf{v} \in \mathbb{R}^4$ with unit L_3 norm i.e., $\|\mathbf{v}\|_3 = 1$.

1	0	0	0

c. A vector $\mathbf{v} \in \mathbb{R}^4$ with $\|\mathbf{v}\|_2 \leq 1$ and $\|\mathbf{v} - \mathbf{u}\|_2 \geq 2$ where $\mathbf{u} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^4$.

$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

d. A vector $\mathbf{v} \in \mathbb{R}^4$ such that $\mathbf{v} \neq \mathbf{0}$ and whose L_2 norm is half its L_1 norm i.e., $\|\mathbf{v}\|_2 = \frac{1}{2}\|\mathbf{v}\|_1$.

$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$