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where \( \gamma > 0 \) is a pre-specified margin. For standard Perceptron, \( \gamma = 0 \)

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Support Vector Machine (SVM)

- Learns a hyperplane such that the positive and negative class training examples are as far away as possible from it (ensures good generalization)

SVMs can also learn nonlinear decision boundaries using kernels (though the idea of kernels is not specific to SVMs and is more generally applicable).

Reason behind the name “Support Vector Machine”? SVM finds the most important examples (called “support vectors”) in the training data. These examples also “balance” the margin boundaries (hence called “support”). Also, even if we throw away the remaining training data and re-learn the SVM classifier, we’ll get the same hyperplane.
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Suppose there exists a hyperplane $w^T x + b = 0$ such that

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- Also note that $\min_{1 \leq n \leq N} |\mathbf{w}^T \mathbf{x}_n + b| = 1$
Learning a Maximum Margin Hyperplane

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Want the hyperplane $(\mathbf{w}, b)$ to have the largest possible margin
Large Margin = Good Generalization

- Large margins intuitively mean good generalization

Large margin $\gamma \propto \|w\|$

Large margin $\Rightarrow$ small $\|w\|$, i.e., small $\ell_2$ norm of $w$

Small $\|w\| \Rightarrow$ regularized/simple solutions ($w$'s don't become too large)

Recall our discussion of regularization.

Simple solutions $\Rightarrow$ good generalization on test data

Want to see an even more formal justification? :-)

Wait until we cover Learning Theory!
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Machine Learning (CS771A)
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Hard-Margin SVM

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- The objective for hard-margin SVM

$$\min_{w, b} f(w, b) = \frac{||w||^2}{2}$$

subject to $y_n(w^T x_n + b) \geq 1, \quad n = 1, \ldots, N$
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- Thus the hard-margin SVM minimizes a convex objective function which is a Quadratic Program (QP) with $N$ linear inequality constraints
Soft-Margin SVM (More Commonly Used)

- Allow some training examples to fall **within the margin region**, or be even **misclassified** (i.e., fall on the wrong side). Preferable **if training data is noisy**
Soft-Margin SVM (More Commonly Used)

- Allow some training examples to fall **within** the margin region, or be even **misclassified** (i.e., fall on the wrong side). Preferable if training data is noisy.

- Each training example \((x_n, y_n)\) given a “slack” \(\xi_n \geq 0\) (distance by which it “violates” the margin). If \(\xi_n > 1\) then \(x_n\) is totally on the wrong side.
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  - Basically, we want a soft-margin condition: \(y_n (w^T x_n + b) \geq 1 - \xi_n, \quad \xi_n \geq 0\)
Soft-Margin SVM (More Commonly Used)

- Goal: Maximize the margin, while also minimizing the sum of slacks (don't want too many training examples violating the margin condition)

The primal objective for soft-margin SVM can thus be written as

\[
\min_w, b, \xi \quad f(w, b, \xi) = ||w||^2 + C \sum_{n=1}^{N} \xi_n
\]

subject to constraints

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Thus the soft-margin SVM also minimizes a convex objective function which is a Quadratic Program (QP) with \(2N\) linear inequality constraints.

Param.

- \(C\) controls the trade-off between large margin vs small training error
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Objective for the hard-margin SVM (unknowns are $w$ and $b$)

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In either case, we have to solve constrained, convex optimization problem
Brief Detour: Solving Constrained Optimization Problems
Constrained Optimization via Lagrangian

Consider optimizing the following objective, subject to some constraints

$$\min_w f(w)$$

s.t. 

$$g_n(w) \leq 0, \quad n = 1, \ldots, N$$

$$h_m(w) = 0, \quad m = 1, \ldots, M$$

Introduce Lagrange multipliers $$\alpha = \{\alpha_n\}^N_{n=1}, \alpha_n \geq 0,$$ and $$\beta = \{\beta_m\}^M_{m=1},$$ one for each constraint, and construct the following Lagrangian

$$L(w, \alpha, \beta) = f(w) + \sum_{n=1}^{N} \alpha_n g_n(w) + \sum_{m=1}^{M} \beta_m h_m(w)$$

Consider $$L^P(w) = \max \alpha, \beta L(w, \alpha, \beta).$$ Note that

$$L^P(w) = \infty$$ if $$w$$ violates any of the constraints ($$g$$'s or $$h$$'s)

$$L^P(w) = f(w)$$ if $$w$$ satisfies all the constraints ($$g$$'s and $$h$$'s)

Thus $$\min_w L^P(w) = \min_w \max_{\alpha \geq 0, \beta} L(w, \alpha, \beta)$$ solves the same problem as the original problem and will have the same solution. For convex $$f, g, h,$$ the order of min and max is interchangeable.

Karush-Kuhn-Tucker (KKT) Conditions: At the optimal solution, $$\alpha_n g_n(w) = 0$$ (note the max $$\alpha$$).
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Solving Hard-Margin SVM
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- The hard-margin SVM optimization problem is:

\[
\begin{align*}
\min_{w,b} & \quad f(w, b) = \frac{||w||^2}{2} \\
\text{subject to} & \quad 1 - y_n(w^T x_n + b) \leq 0, \quad n = 1, \ldots, N
\end{align*}
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- A constrained optimization problem. Can solve using Lagrange’s method
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- Introduce Lagrange Multipliers $$\alpha_n \ (n = \{1, \ldots, N\})$$, one for each constraint, and solve the following Lagrangian:

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- Note: $$\alpha = [\alpha_1, \ldots, \alpha_N]$$ is the vector of Lagrange multipliers
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\min_{w, b} \quad f(w, b) = \frac{||w||^2}{2}
\]
subject to \( 1 - y_n(w^T x_n + b) \leq 0, \quad n = 1, \ldots, N \)

- A constrained optimization problem. Can solve using Lagrange’s method

- Introduce Lagrange Multipliers \( \alpha_n \) \((n = \{1, \ldots, N\})\), one for each constraint, and solve the following Lagrangian:

\[
\min_{w, b} \quad \max_{\alpha \geq 0} \quad \mathcal{L}(w, b, \alpha) = \frac{||w||^2}{2} + \sum_{n=1}^{N} \alpha_n \{1 - y_n(w^T x_n + b)\}
\]

- Note: \( \alpha = [\alpha_1, \ldots, \alpha_N] \) is the vector of Lagrange multipliers

- We will solve this Lagrangian by solving a dual problem (eliminate \( w \) and \( b \) and solve for the “dual variables” \( \alpha \))
Solving Hard-Margin SVM

The original Lagrangian is

\[
\min_{w, b} \max_{\alpha \geq 0} \mathcal{L}(w, b, \alpha) = \frac{w^T w}{2} + \sum_{n=1}^{N} \alpha_n \{1 - y_n (w^T x_n + b)\}
\]

Take (partial) derivatives of \(\mathcal{L}\) w.r.t. \(w\), \(b\) and set them to zero

\[
\frac{\partial \mathcal{L}}{\partial w} = 0 \Rightarrow w = \sum_{n=1}^{N} \alpha_n y_n x_n
\]

\[
\frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow \sum_{n=1}^{N} \alpha_n y_n = 0
\]

Important: Note the form of the solution \(w\) - it is simply a weighted sum of all the training inputs \(x_1, \ldots, x_N\) (and \(\alpha_n\) is like the "importance" of \(x_n\)).

Substituting \(w = \sum_{n=1}^{N} \alpha_n y_n x_n\) in Lagrangian and also using \(\sum_{n=1}^{N} \alpha_n y_n = 0\)

\[
\max_{\alpha \geq 0} \mathcal{L}(\alpha) = \sum_{n=1}^{N} \alpha_n - \frac{1}{2} \sum_{m, n=1}^{N} \alpha_m \alpha_n y_m y_n (x_m^T x_n + b)
\]
Solving Hard-Margin SVM

- The original Lagrangian is

\[
L(w, b, \alpha) = \frac{w^T w}{2} + \sum_{n=1}^{N} \alpha_n \{1 - y_n(w^T x_n + b)\}
\]

- Take (partial) derivatives of \(L\) w.r.t. \(w\), \(b\) and set them to zero

\[
\frac{\partial L}{\partial w} = 0 \implies w = \sum_{n=1}^{N} \alpha_n y_n x_n \quad \frac{\partial L}{\partial b} = 0 \implies \sum_{n=1}^{N} \alpha_n y_n = 0
\]
Solving Hard-Margin SVM

The original Lagrangian is

\[
\min_{w, b} \max_{\alpha \geq 0} L(w, b, \alpha) = \frac{w^T w}{2} + \sum_{n=1}^{N} \alpha_n \{1 - y_n (w^T x_n + b)\}
\]

Take (partial) derivatives of \( L \) w.r.t. \( w \), \( b \) and set them to zero

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\frac{\partial L}{\partial w} = 0 \Rightarrow w = \sum_{n=1}^{N} \alpha_n y_n x_n \\
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\]

Important: Note the form of the solution \( w \) - it is simply a weighted sum of all the training inputs \( x_1, \ldots, x_N \) (and \( \alpha_n \) is like the “importance” of \( x_n \))

Substituting \( w = \sum_{n=1}^{N} \alpha_n y_n x_n \) in Lagrangian and also using \( \sum_{n=1}^{N} \alpha_n y_n = 0 \)

\[
\max_{\alpha \geq 0} L_D(\alpha) = \sum_{n=1}^{N} \alpha_n - \frac{1}{2} \sum_{m, n=1}^{N} \alpha_m \alpha_n y_m y_n (x_m^T x_n) \quad \text{s.t.} \quad \sum_{n=1}^{N} \alpha_n y_n = 0
\]
Solving Hard-Margin SVM

- Can write the objective more compactly in vector/matrix form as

\[
\max_{\alpha \geq 0} \mathcal{L}_D(\alpha) = \alpha^\top 1 - \frac{1}{2} \alpha^\top G \alpha \quad \text{s.t.} \quad \sum_{n=1}^{N} \alpha_n y_n = 0
\]

where \( G \) is an \( N \times N \) matrix with \( G_{mn} = y_m y_n x_m^\top x_n \), and \( 1 \) is a vector of 1s

† If interested in more details of the solver, see: “Support Vector Machine Solvers” by Bottou and Lin
Can write the objective more compactly in vector/matrix form as

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**Good news:** This is maximizing a concave function (or minimizing a convex function - verify that the Hessian is \(G\), which is p.s.d.). Note that our original primal SVM objective was also convex

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Solving Hard-Margin SVM

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- Can solve† the above objective function for \(\alpha\) using various methods, e.g.,
  - Treating the objective as a **Quadratic Program** (QP) and running some off-the-shelf QP solver such as quadprog (MATLAB), CVXOPT, CPLEX, etc.

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Solving Hard-Margin SVM

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- **Important:** Inputs \( x \)'s only appear as inner products (helps to "kernelize")

- Can solve\(^\dagger\) the above objective function for \( \alpha \) using various methods, e.g.,

  - Treating the objective as a Quadratic Program (QP) and running some off-the-shelf QP solver such as quadprog (MATLAB), CVXOPT, CPLEX, etc.
  
  - Using (projected) gradient methods (projection needed because the \( \alpha \)'s are constrained). Gradient methods will usually be much faster than QP methods.

\(^\dagger\) If interested in more details of the solver, see: “Support Vector Machine Solvers” by Bottou and Lin
Once we have the $\alpha_n$'s, $\mathbf{w}$ and $b$ can be computed as:

$$\mathbf{w} = \sum_{n=1}^{N} \alpha_n y_n \mathbf{x}_n$$

$$b = -\frac{1}{2} \left( \min_{n:y_n=+1} \mathbf{w}^T \mathbf{x}_n + \max_{n:y_n=-1} \mathbf{w}^T \mathbf{x}_n \right)$$
Hard-Margin SVM: The Solution

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- **A nice property:** Most $\alpha_n$'s in the solution will be zero (sparse solution)

- Reason: Karush-Kuhn-Tucker (KKT) conditions
Hard-Margin SVM: The Solution

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Hard-Margin SVM: The Solution

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- $\alpha_n$ is non-zero only if $x_n$
Once we have the $\alpha_n$’s, $w$ and $b$ can be computed as:

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For the optimal $\alpha_n$’s

$$\alpha_n \{1 - y_n(w^T x_n + b)\} = 0$$

$\alpha_n$ is non-zero only if $x_n$ lies on one of the two margin boundaries, i.e., for which $y_n(w^T x_n + b) = 1$
Once we have the $\alpha_n$’s, $w$ and $b$ can be computed as:

$$ w = \sum_{n=1}^{N} \alpha_n y_n x_n $$

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$$ \alpha_n \{1 - y_n (w^T x_n + b)\} = 0 $$

- $\alpha_n$ is non-zero only if $x_n$ lies on one of the two margin boundaries, i.e., for which $y_n (w^T x_n + b) = 1$
- These examples are called support vectors
Hard-Margin SVM: The Solution

Once we have the $\alpha_n$'s, $w$ and $b$ can be computed as:

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For the optimal $\alpha_n$'s

$$ \alpha_n \{1 - y_n(w^T x_n + b)\} = 0 $$

$\alpha_n$ is non-zero only if $x_n$ lies on one of the two margin boundaries, i.e., for which $y_n(w^T x_n + b) = 1$

These examples are called support vectors

Recall the support vectors “support” the margin boundaries
Solving Soft-Margin SVM
Recall the soft-margin SVM optimization problem:

\[
\min_{w, b, \xi} f(w, b, \xi) = \frac{||w||^2}{2} + C \sum_{n=1}^{N} \xi_n \\
\text{subject to } 1 \leq y_n(w^T x_n + b) + \xi_n, \quad -\xi_n \leq 0 \quad n = 1, \ldots, N
\]

Note: \( \xi = [\xi_1, \ldots, \xi_N] \) is the vector of slack variables
Recall the soft-margin SVM optimization problem:

\[
\min_{w, b, \xi} f(w, b, \xi) = \frac{||w||^2}{2} + C \sum_{n=1}^{N} \xi_n \\
\text{subject to } 1 \leq y_n(w^T x_n + b) + \xi_n, \quad -\xi_n \leq 0 \quad n = 1, \ldots, N
\]

Note: \( \xi = [\xi_1, \ldots, \xi_N] \) is the vector of slack variables

Introduce Lagrange Multipliers \( \alpha_n, \beta_n \) \((n = \{1, \ldots, N\})\), for constraints, and solve the Lagrangian:

\[
\min_{w, b, \xi} \max_{\alpha \geq 0, \beta \geq 0} \mathcal{L}(w, b, \xi, \alpha, \beta) = \frac{||w||^2}{2} + C \sum_{n=1}^{N} \xi_n + \sum_{n=1}^{N} \alpha_n \{1 - y_n(w^T x_n + b) - \xi_n\} - \sum_{n=1}^{N} \beta_n \xi_n
\]
Solving Soft-Margin SVM

- Recall the soft-margin SVM optimization problem:

\[
\min_{w,b,\xi} f(w, b, \xi) = \frac{||w||^2}{2} + C \sum_{n=1}^{N} \xi_n
\]

subject to \[ 1 \leq y_n(w^T x_n + b) + \xi_n, \quad -\xi_n \leq 0 \quad n = 1, \ldots, N \]

Note: \( \xi = [\xi_1, \ldots, \xi_N] \) is the vector of slack variables

- Introduce Lagrange Multipliers \( \alpha_n, \beta_n \) \( (n = \{1, \ldots, N\}) \), for constraints, and solve the Lagrangian:

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\]

Note: The terms in red above were not present in the hard-margin SVM
Solving Soft-Margin SVM

- Recall the soft-margin SVM optimization problem:

\[
\begin{align*}
\min_{w, b, \xi} & \quad f(w, b, \xi) = \frac{||w||^2}{2} + C \sum_{n=1}^{N} \xi_n \\
\text{subject to} & \quad 1 \leq y_n(w^T x_n + b) + \xi_n, \quad -\xi_n \leq 0 \quad n = 1, \ldots, N
\end{align*}
\]

Note: \(\xi = [\xi_1, \ldots, \xi_N]\) is the vector of slack variables

- Introduce Lagrange Multipliers \(\alpha_n, \beta_n\) \((n = \{1, \ldots, N\})\), for constraints, and solve the Lagrangian:

\[
\begin{align*}
\min_{w, b, \xi} \max_{\alpha \geq 0, \beta \geq 0} \quad L(w, b, \xi, \alpha, \beta) = \frac{||w||^2}{2} + C \sum_{n=1}^{N} \xi_n + \sum_{n=1}^{N} \alpha_n \{1 - y_n(w^T x_n + b) - \xi_n\} - \sum_{n=1}^{N} \beta_n \xi_n
\end{align*}
\]

Note: The terms in red above were not present in the hard-margin SVM

- Two sets of dual variables \(\alpha = [\alpha_1, \ldots, \alpha_N]\) and \(\beta = [\beta_1, \ldots, \beta_N]\). We'll eliminate the primal variables \(w, b, \xi\) to get dual problem containing the dual variables (just like in the hard margin case)
Solving Soft-Margin SVM

- The Lagrangian problem to solve

\[
\min_{w, b, \xi} \max_{\alpha \geq 0, \beta \geq 0} \mathcal{L}(w, b, \xi, \alpha, \beta) = \frac{w^T w}{2} + C \sum_{n=1}^{N} \xi_n + \sum_{n=1}^{N} \alpha_n \{1 - y_n(w^T x_n + b) - \xi_n\} - \sum_{n=1}^{N} \beta_n \xi_n
\]
Solving Soft-Margin SVM

- The Lagrangian problem to solve

\[
\min_{w,b,\xi} \; \max_{\alpha \geq 0, \beta \geq 0} \quad \mathcal{L}(w, b, \xi, \alpha, \beta) = \frac{w^T w}{2} + C \sum_{n=1}^{N} \xi_n + \sum_{n=1}^{N} \alpha_n \{1 - y_n (w^T x_n + b) - \xi_n\} - \sum_{n=1}^{N} \beta_n \xi_n
\]

- Take (partial) derivatives of \( \mathcal{L} \) w.r.t. \( w, b, \xi_n \) and set them to zero

\[
\frac{\partial \mathcal{L}}{\partial w} = 0 \quad \Rightarrow \quad w = \sum_{n=1}^{N} \alpha_n y_n x_n,
\quad \frac{\partial \mathcal{L}}{\partial b} = 0 \quad \Rightarrow \quad \sum_{n=1}^{N} \alpha_n y_n = 0,
\quad \frac{\partial \mathcal{L}}{\partial \xi_n} = 0 \quad \Rightarrow \quad C - \alpha_n - \beta_n = 0
\]
Solving Soft-Margin SVM

• The Lagrangian problem to solve

\[
\min_{w,b,\xi} \max_{\alpha,\beta \geq 0} \mathcal{L}(w,b,\xi,\alpha,\beta) = \frac{w^T w}{2} + C \sum_{n=1}^{N} \xi_n + \sum_{n=1}^{N} \alpha_n \{1 - y_n (w^T x_n + b) - \xi_n\} - \sum_{n=1}^{N} \beta_n \xi_n
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• Take (partial) derivatives of \( \mathcal{L} \) w.r.t. \( w, b, \xi_n \) and set them to zero

\[
\frac{\partial \mathcal{L}}{\partial w} = 0 \Rightarrow w = \sum_{n=1}^{N} \alpha_n y_n x_n, \quad \frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow \sum_{n=1}^{N} \alpha_n y_n = 0, \quad \frac{\partial \mathcal{L}}{\partial \xi_n} = 0 \Rightarrow C - \alpha_n - \beta_n = 0
\]

• Note: Solution of \( w \) again has the same form as in the hard-margin case (weighted sum of all inputs with \( \alpha_n \) being the importance of input \( x_n \))
Solving Soft-Margin SVM

- The Lagrangian problem to solve

\[
\min_{w,b,\xi} \max_{\alpha \geq 0, \beta \geq 0} \mathcal{L}(w, b, \xi, \alpha, \beta) = \frac{w^T w}{2} + C \sum_{n=1}^{N} \xi_n + \sum_{n=1}^{N} \alpha_n \{1 - y_n (w^T x_n + b) - \xi_n\} - \sum_{n=1}^{N} \beta_n \xi_n
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\frac{\partial \mathcal{L}}{\partial w} = 0 \Rightarrow w = \sum_{n=1}^{N} \alpha_n y_n x_n, \quad \frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow \sum_{n=1}^{N} \alpha_n y_n = 0, \quad \frac{\partial \mathcal{L}}{\partial \xi_n} = 0 \Rightarrow C - \alpha_n - \beta_n = 0
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- Note: Using \( C - \alpha_n - \beta_n = 0 \) and \( \beta_n \geq 0 \Rightarrow \alpha_n \leq C \) (recall that, for the hard-margin case, \( \alpha \geq 0 \))
Solving Soft-Margin SVM

The Lagrangian problem to solve

\[
\min_{w, b, \xi} \max_{\alpha \geq 0, \beta \geq 0} \mathcal{L}(w, b, \xi, \alpha, \beta) = \frac{w^T w}{2} + C \sum_{n=1}^{N} \xi_n + \sum_{n=1}^{N} \alpha_n \{1 - y_n(w^T x_n + b) - \xi_n\} - \sum_{n=1}^{N} \beta_n \xi_n
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Take (partial) derivatives of \(\mathcal{L}\) w.r.t. \(w, b, \xi_n\) and set them to zero

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\frac{\partial \mathcal{L}}{\partial w} = 0 \Rightarrow w = \sum_{n=1}^{N} \alpha_n y_n x_n, \quad \frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow \sum_{n=1}^{N} \alpha_n y_n = 0, \quad \frac{\partial \mathcal{L}}{\partial \xi_n} = 0 \Rightarrow C - \alpha_n - \beta_n = 0
\]

Note: Solution of \(w\) again has the same form as in the hard-margin case (weighted sum of all inputs with \(\alpha_n\) being the importance of input \(x_n\))

Note: Using \(C - \alpha_n - \beta_n = 0\) and \(\beta_n \geq 0 \Rightarrow \alpha_n \leq C\) (recall that, for the hard-margin case, \(\alpha \geq 0\))

Substituting these in the Lagrangian \(\mathcal{L}\) gives the Dual problem

\[
\max_{\alpha \leq C, \beta \geq 0} \mathcal{L}_D(\alpha, \beta) = \sum_{n=1}^{N} \alpha_n - \frac{1}{2} \sum_{m,n=1}^{N} \alpha_m \alpha_n y_m y_n (x_m^T x_n) \quad \text{s.t.} \quad \sum_{n=1}^{N} \alpha_n y_n = 0
\]
Solving Soft-Margin SVM

- Interestingly, the dual variables $\beta$ don’t appear in the objective!

\[ \text{max} \quad \alpha \leq C \quad L D(\alpha) = \alpha^\top 1 - \frac{1}{2} \alpha^\top G \alpha \]
\[ \text{s.t.} \quad \sum_{n=1}^{N} \alpha_n y_n = 0 \]

Where $G$ is an $N \times N$ matrix with $G_{mn} = y_m y_n x^\top_m x_n$, and $1$ is a vector of 1s.

Like hard-margin case, solving the dual requires concave maximization (or convex minimization). Can be solved the same way as hard-margin SVM (except that $\alpha \leq C$).

Can solve for $\alpha$ using QP solvers or (projected) gradient methods.

Given $\alpha$, the solution for $w$, $b$ has the same form as hard-margin case.

Note: $\alpha$ is again sparse. Nonzero $\alpha_n$’s correspond to the support vectors.

† If interested in more details of the solver, see: “Support Vector Machine Solvers” by Bottou and Lin.
Solving Soft-Margin SVM

- Interestingly, the dual variables $\beta$ don’t appear in the objective!
- Just like the hard-margin case, we can write the dual more compactly as

$$\max_{\alpha \leq C} \mathcal{L}_D(\alpha) = \alpha^\top 1 - \frac{1}{2} \alpha^\top G \alpha \quad \text{s.t.} \quad \sum_{n=1}^{N} \alpha_n y_n = 0$$

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Solving Soft-Margin SVM

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$$\max_{\alpha \leq C} L_D(\alpha) = \alpha^T 1 - \frac{1}{2} \alpha^T G \alpha \quad \text{s.t.} \quad \sum_{n=1}^{N} \alpha_n y_n = 0$$

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Solving Soft-Margin SVM

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Support Vectors in Soft-Margin SVM

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Recall the final dual objectives for hard-margin and soft-margin SVM

**Hard-Margin SVM:**

\[ \max_{\alpha \geq 0} \mathcal{L}_D(\alpha) = \alpha^\top \mathbf{1} - \frac{1}{2} \alpha^\top \mathbf{G} \alpha \]

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SVMs via Dual Formulation: Some Comments

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- A lot of work\(^\dagger\) has gone into speeding up optimization in these settings

\(^\dagger\) See: “Support Vector Machine Solvers” by Bottou and Lin
Convex Hull Interpretation\(^\dagger\): Solving the SVM dual is equivalent to finding the shortest line connecting the convex hulls of both classes (the SVM’s hyperplane will be the perpendicular bisector of this line)

\(^\dagger\) See: “Duality and Geometry in SVM Classifiers” by Bennett and Bredensteiner
Recall, we want for each training example: \( y_n(w^T x_n + b) \geq 1 - \xi_n \)
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- Perceptron, SVM, Logistic Reg., all minimize convex approximations of the 0-1 loss (optimizing which is NP-hard; moreover it’s non-convex/non-smooth)
SVM: Some Notes

- A hugely (perhaps the most!) popular classification algorithm
- Reasonably mature, highly optimized SVM softwares freely available (perhaps the reason why it is more popular than various other competing algorithms)
  - Some popular ones: libSVM, LIBLINEAR, SVMStruct, Vowpal Wabbit, etc.
- Lots of work on scaling up SVMs† (both large \( N \) and large \( D \))
- Extensions beyond binary classification (e.g., multiclass, structured outputs)
- Can even be used for regression problems (Support Vector Regression)
- Nonlinear extensions possible via kernels

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