Introduction to Learning Theory

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Machine Learning (CS771A)

Oct 24, 2016

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"Theory is the first term in the Taylor series expansion of Practice" - T. Cover

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    - We want to know how much worse it is..
    - .. without doing experiments

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- Since $K \leq |\mathcal{H}|$, K can be replaced by the size of set $\mathcal{H}$

$$P_{\mathcal{D} \sim P^N}(\exists h : \text{“}h \text{ is bad”}) \leq |\mathcal{H}|(1 - \epsilon)^N \quad \text{(Uniform Convergence)}$$

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PAC Learnability and Efficient PAC Learnability

- **Definition:** An algorithm $\mathcal{A}$ is an (ϵ, δ)-PAC learning algorithm if, for all distributions $\mathcal{D}$: given samples from $\mathcal{D}$, the probability that it returns a "bad hypothesis" h is at most δ, where a "bad" hypothesis is one with test error rate more than ϵ on $\mathcal{D}$.

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    - Note: a similar notion of an efficient (ϵ, δ)-PAC learning algorithm holds in terms of the number of training examples required (polynomial in $\frac{1}{\epsilon}$ and $\frac{1}{\delta}$)

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- Using the union bound, we have:

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- Thus $\log_2 H_k \leq (k-1)\log_2 D + 2k - 1$ (linear in k)

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  - .. enters the Vapnik-Chervonenkis dimension (VC dimension)
  - VC dimension: a measure of the complexity of a hypothesis class

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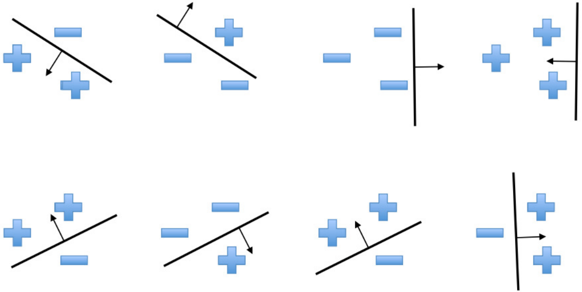

- Figure above: 3 points in 2D, $\mathcal{H}$: set of linear classifiers

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The VC dimension of $\mathcal{H}$, *in that input space*, is the maximum d we can choose so that we always succeed in the game

VC Dimension: The Shattering Game

The concept of shattering is used to define the VC dimension of hypothesis classes

Consider the following shattering game between us and an adversary

- We choose d points in an input space, positioned however we want
- Adversary labels these d points
- We find a hypothesis $h \in \mathcal{H}$ that separates the points
- Note: Shattering just one configuration of d points is enough to win

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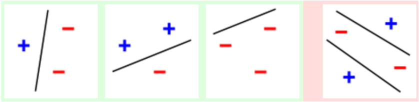

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VC dimension intuition: How many points the hypothesis class can "memorize"

Using VC Dimension in Generalization Bounds

Recall the PAC based Generalization Bound

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Having fewer features is better (since it means smaller VC dimension)

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Large $\gamma \Rightarrow$ small VC dim. $\Rightarrow$ small complexity of $\mathcal{H}_\gamma \Rightarrow$ good generalization

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  - But even loose bounds are often useful for understanding the basic properties of learning models/algorithms