Introduction to Generative Models

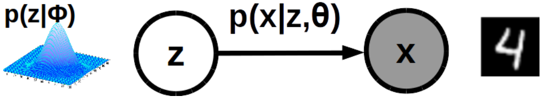
Piyush Rai

Machine Learning (CS771A)

Sept 23, 2016

Generative Model

- Defines a probabilistic way that could have “generated” the data



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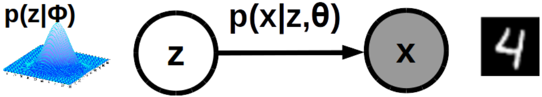

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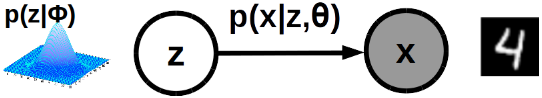

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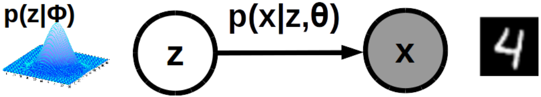

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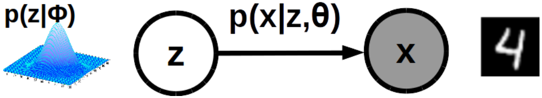

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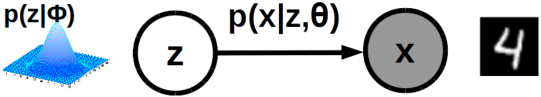

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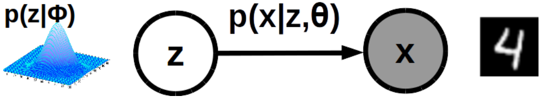

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- $\{\theta, \phi\}$ are the unknown model parameters
- The goal will be to learn $\{\theta, \phi\}$ and $\boldsymbol{z}_n$'s, given the observed data

"Generative Story" of Data

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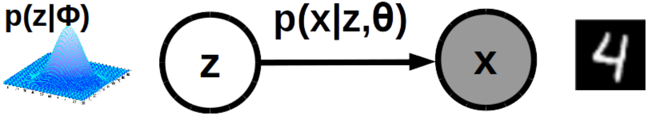

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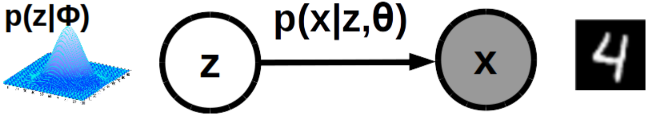

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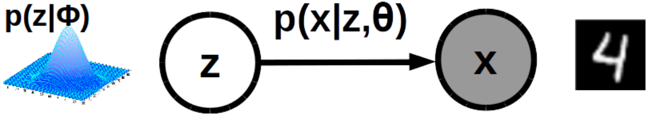

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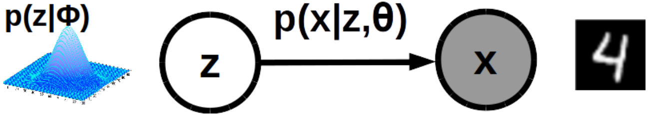

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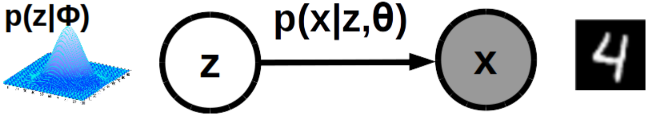

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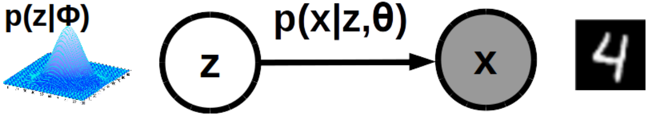

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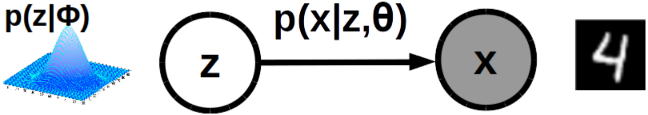

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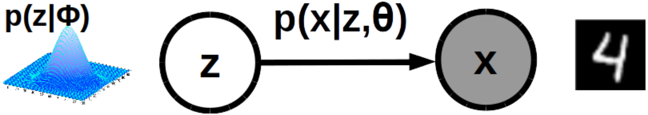

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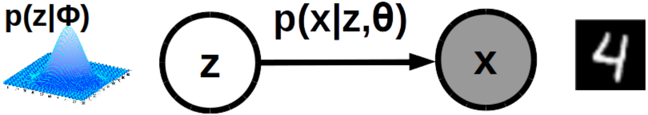

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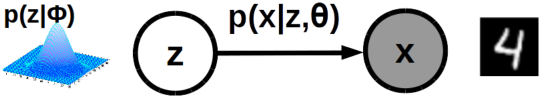
  - Usually it’s possible to infer the global vars from local vars (or vice-versa)

Why Generative Models?

- A proper, probabilistic way to think about the data generation process



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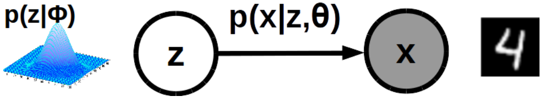

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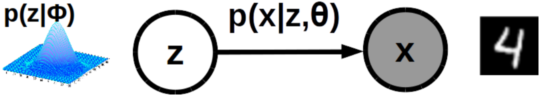

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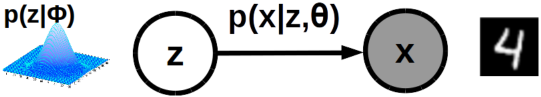

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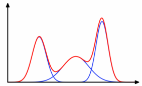
- Allows handling missing data (by treating missing data also as latent variable)

Hallucinated faces pic courtesy: `http://torch.ch/blog/2015/11/13/gan.html`

Some "Canonical" Generative Models

- Mixture model (used in clustering and probability density estimation)



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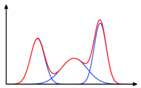
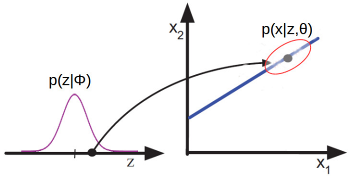


- Latent factor model (used in dimensionality reduction)



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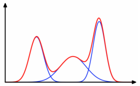

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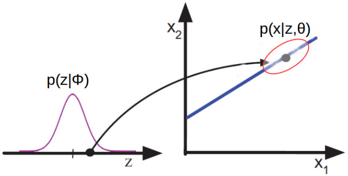

- Can even combine these (e.g., mixture of latent factor models)

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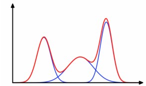

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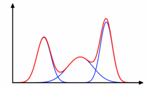

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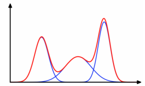

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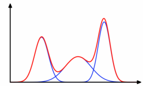

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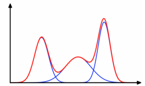

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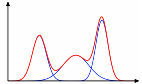

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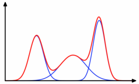

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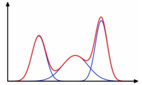

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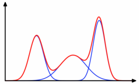

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- Mixture models can model complex distributions as superposition of simpler distributions (can be used for density estimation, as well as clustering).

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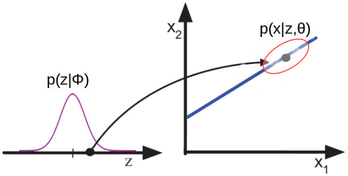

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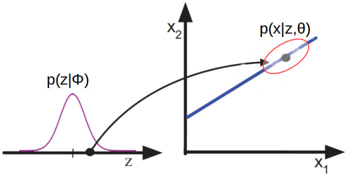

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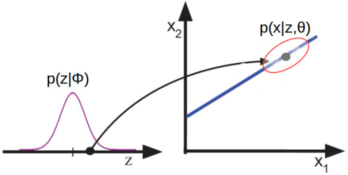

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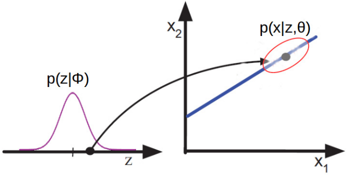

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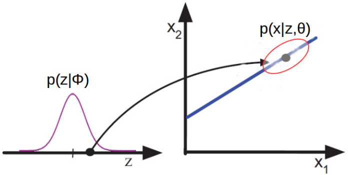

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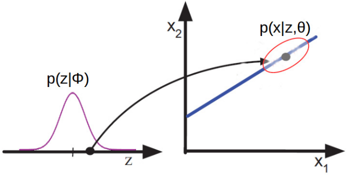

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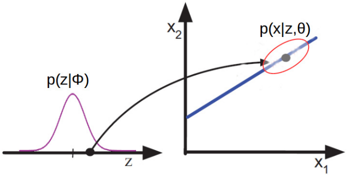

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- Many recent advances in generative models (e.g., deep generative models, generative adversarial networks, etc) are based on these basic principles

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  - Basic idea in EM: Instead of summing over all possibilities of $\boldsymbol{z}$, make a "guess" $\tilde{\boldsymbol{z}}$ and maximize $\log p(\boldsymbol{x}, \tilde{\boldsymbol{z}}|\theta,\phi)$ w.r.t. θ, ϕ to learn θ, ϕ. Use these values of θ, ϕ to refine your guess $\tilde{\boldsymbol{z}}$ and repeat until convergence.

Gaussian Mixture Model (GMM)

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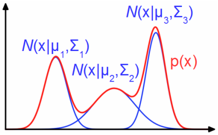

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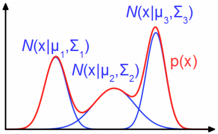

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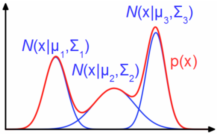

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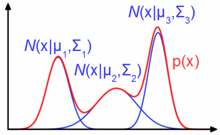

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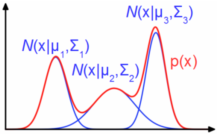

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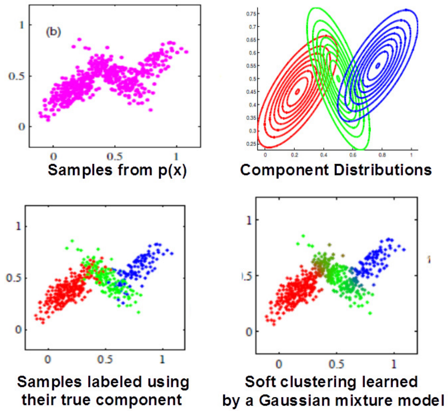
- GMM in many ways improves over K-means clustering

GMM Clustering: Pictorially

Some synthetically generated data (top-left) generated from a mixture of 3 overlapping Gaussians (top-right).



Notice the "mixed" colored points in the overlapping regions in the final clustering

Next Class

- GMM in more detail. Extensions of GMM.
- Parameter estimation in GMM
- The Expectation Maximization (EM) algorithm