# Algebra powers computation

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[*Thanks to the artists*]

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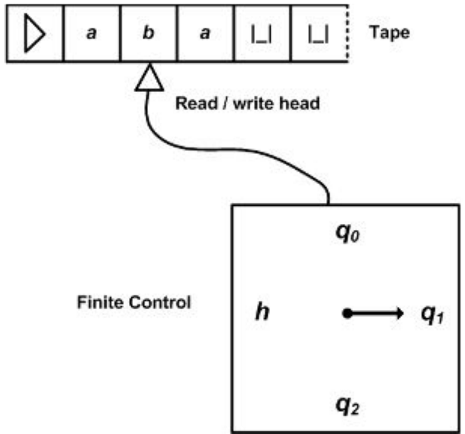
# What's computing?

- Alan Turing (1936) postulated a simple, most general, mathematical model for computing – **Turing machine** (TM).

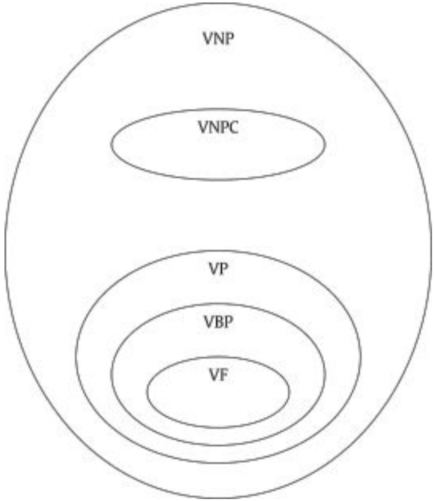
- **Algorithm** = TM is very much like a computer *program*.
  - TM is a real computer – highly iterative & trivial steps.

- How about an **electronic** circuit?
  - Algebraically, it's a neater model to capture real computation.

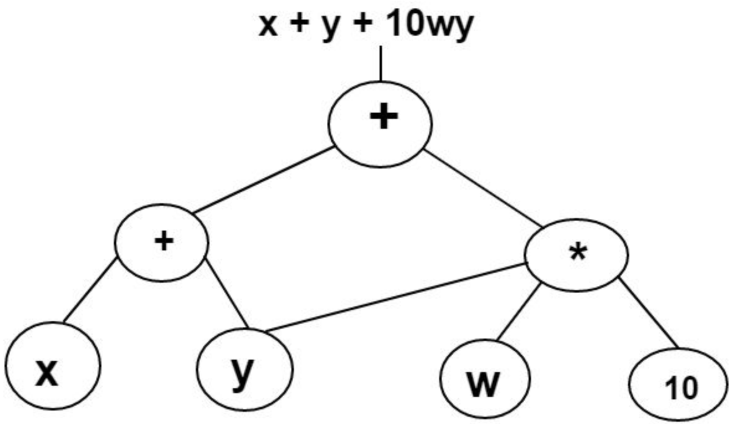



Turing (1912-1954)

# Valiant: Algebraic circuits

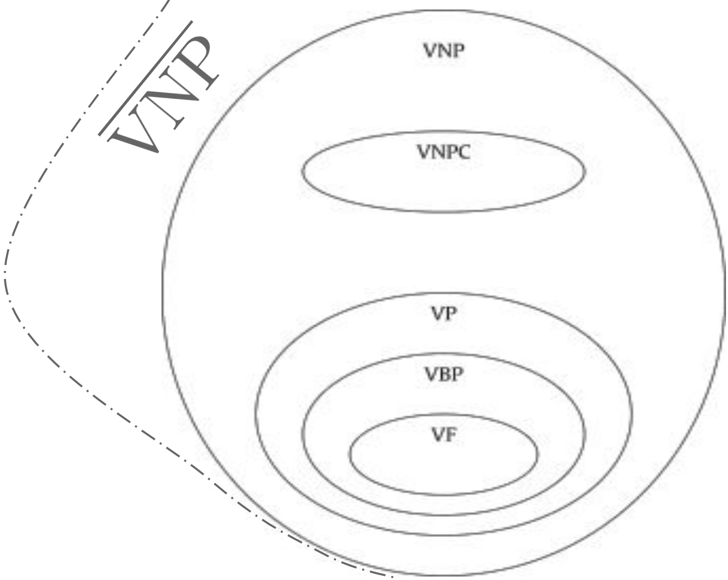



- Valiant (1977) formalized computation & resources using algebraic circuits.
  - Giving birth to his VP ≠ VNP question.
  - Or, the algebraic hardness question!
- Algebraic circuit has constants/variables, size, depth.

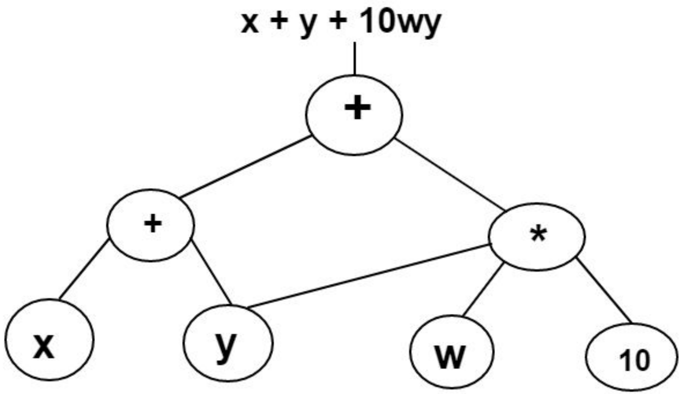



Leslie Valiant (1949-)

# Valiant: Algebraic circuits




- My work: Study circuit problems and their properties.
  - Develop the relevant mathematics.




- De-fictionalize the above picture!
  - Progress has been impressive.
  - Withstands AI hype.

# Zero or nonzero: PIT

$$(a_1^2+a_2^2+a_3^2+a_4^2)(b_1^2+b_2^2+b_3^2+b_4^2)$$
$$=(a_1b_1-a_2b_2-a_3b_3-a_4b_4)^2+(a_1b_2+a_2b_1+a_3b_4-a_4b_3)^2$$
$$+(a_1b_3-a_2b_4+a_3b_1+a_4b_2)^2+(a_1b_4+a_2b_3-a_3b_2+a_4b_1)^2.$$

**Euler's identity (1749)**

- Question: Test whether a given circuit is zero.
  - Polynomial identity testing (PIT).

| | | |
|---|---|---|
| 10 | = | $1^2+1^2+2^2+2^2$ |
| 103 | = | $2^2+3^2+3^2+9^2$ |
| 312 | = | $2^2+4^2+6^2+16^2$ |

**Lagrange's four-square theorem (1770)**

- OPEN Qn: Is PIT in deterministic polynomial time?

- Motivates new tools to study algebraic computation.

$$(X+1)^n \equiv X^n + 1 \bmod n \iff n \text{ is ?}$$

# Zero or nonzero: PIT

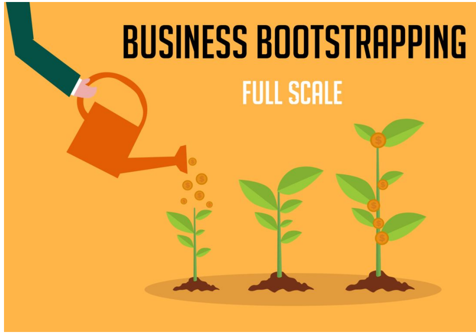
$$(X+1)^n \equiv X^n + 1 \bmod n$$

- Primality testing.
- Blackbox algorithms/
  - Lower bounds (for certain models).
- Incidence-geometry in identities, over all fields.
  - Higher-dimension rank concepts.
- Duality in circuits.
  - Diagonal depth-3 or 4.
- Bootstrapping in circuits.
  - Tiny circuits
  - Sum-of-squares.

Black Box Testing


$$x_1\,, x_2\,, x_1^2 + x_1 x_2 \text{ are dependent.}$$

$$(x_1 + \cdots + x_n)^d\,, \text{ as } f_1(x_1)\cdots f_n(x_n)\,?$$

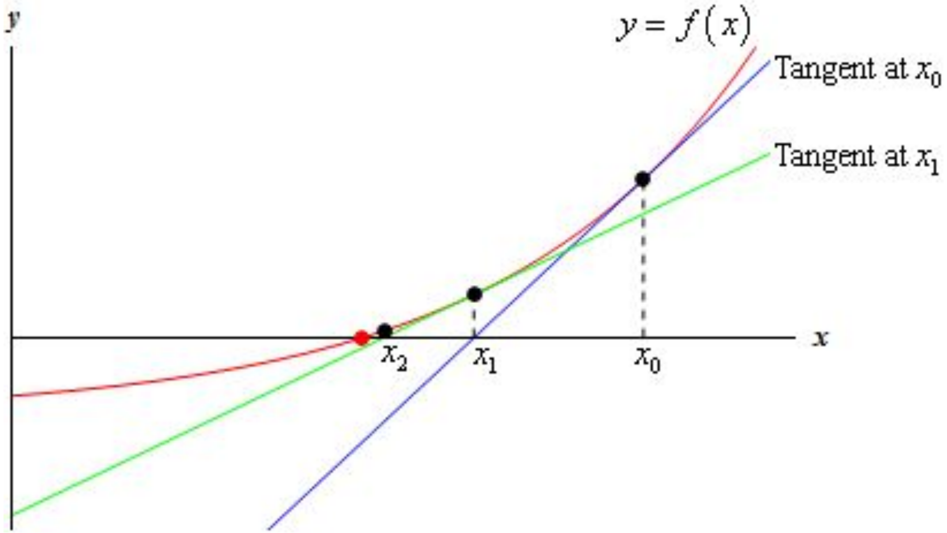



$$f \stackrel{?}{=} f_1(x)^2 + \cdots + f_n(x)^2 \implies ?$$

# Algebraic Algorithms

# Computational Algebra

- ❖ All-roots Newton iteration
  - ➢ Non-simple roots?
  - ➢ N-variate circuit version.




- ❖ Factoring polynomials.
  - ➢ Mod primes, prime-powers, p-adics
  - ➢ Circuit models
  - ➢ Approximative circuits

$$\sqrt{2} = 3 + 1 \times 7 + 2 \times 7^2 + 6 \times 7^3 + \cdots$$

$$x = 0 = x \cdot y - 1 \text{ has } \underline{\text{root}} \; = (\epsilon \to 0, 1/\epsilon \to \infty)$$

**But, has <u>no</u> actual root!**

# Computational Algebra

- Algebraic dependence criteria
- Morphism problems in algebras, graphs
- Roots counting
- Compute Zeta function analogs
  - Infinite information?

$\mathbb{Q}$–dependence of e and $\pi$ ?

$x_1+1\,,x_1+x_2\,,x_1^2+x_1x_2$ are dependent?

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

$x^2 = 0 \bmod p^2$ has $p$ roots

$$F(x\,,y) = 0$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 0$$

with $s \in \mathbb{C}$

**Hardest question on earth since 1859.**

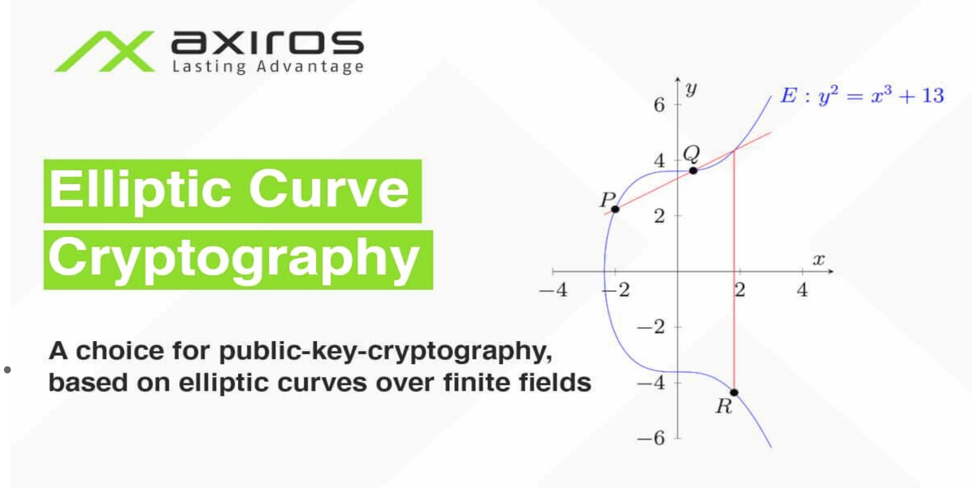
$$P(t) := \sum_{i\geq 0} N_{p^i}(F)/p^{2i}\cdot t^i = ?$$

$$\stackrel{?}{=} 1/(1-t)\ ?$$

Is it always this simple?

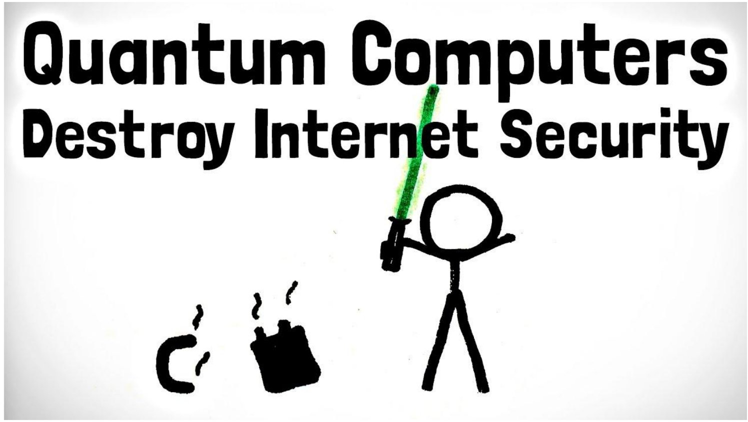
# Engineering

# Feeling insecure?



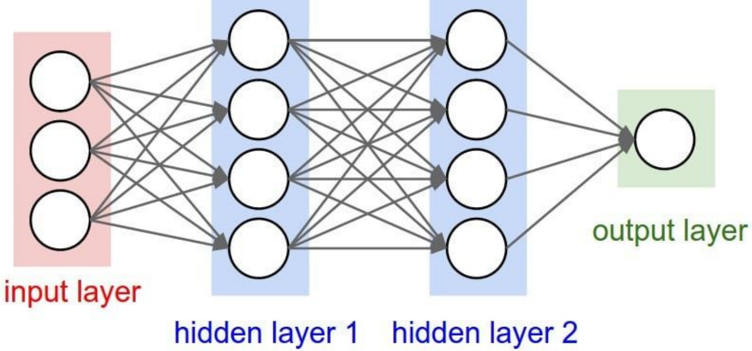

- Cryptography builds on algebra.
  - Number theory
  - Curves, counting, morphism
  - Multivariate systems

- Post-quantum world requires new protocols.
  - Avoid *abelian* groups.
  - Use more complicated geometry,
  - algebra,
  - and lattices.

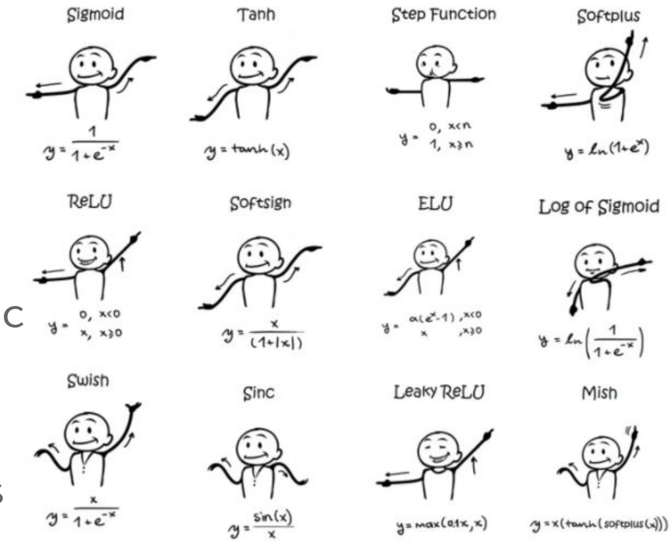



- Inspiration from NP-hard problems?
  - interesting beyond quantum hype

# Practical Learnings




- AI/Machine Learning: Decision-making using circuits.
  - Artificial Neural Networks (ANN).
- ANN is a specialized algebraic circuit.
  - Activation functions are real algebraic




- A Center @IITK to solve practical problems using AI methods.
  - Visit (Center for Developing Intelligent Systems) www.iitk.ac.in/cdis/ www.cse.iitk.ac.in/users/nitin/

THANKS!