# From Hilbert's Entscheidungsproblem to Valiant's counting problem

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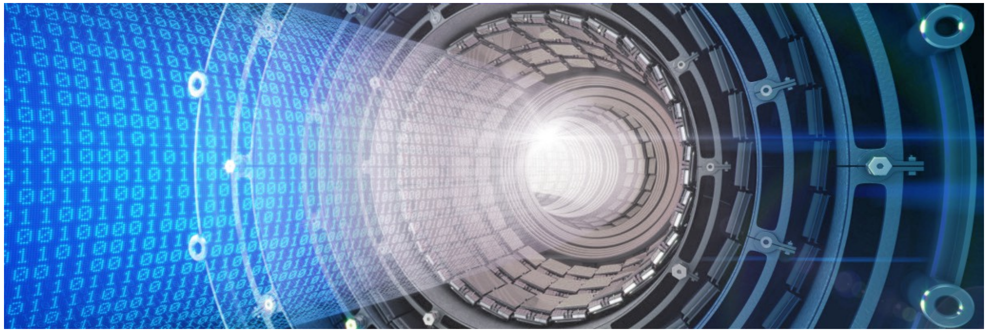


*2014*

# Contents

- Hilbert
- Church & Turing
- Cook & Levin
- Valiant's permanent
- Zero or nonzero
- Fundamental goals

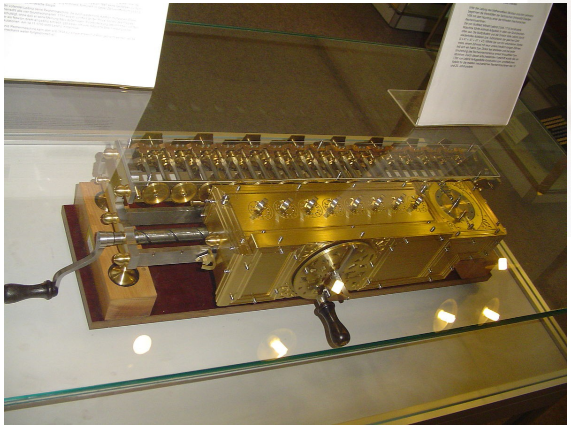
# Hilbert

- Gottfried Leibniz dreamt of building a machine that could check the truth of math statements.
- He was the first to design a machine that could do all the *four* arithmetic operations.
- This led to his optimism that machines might also prove theorems.

  Example 1: Angles of a triangle sum to 180°.

  Example 2: Prime numbers are infinitely many.

Leibniz (1646-1716)

Leibnizrechenmaschine ~1694

# Hilbert

- Leibniz's dream was generalized by Hilbert (1928), who asked for

  "***an <u>algorithm</u> to decide whether a given statement is provable from the axioms using the rules of logic***".

  - Known as the *Entscheidungsproblem*.

Hilbert (1862-1943)

- Like Leibniz, he "believed" that there exists no *undecidable* problem!

- The answer first requires defining 'algorithm'.
  - hence, 'computation' requires a new mathematical framework.

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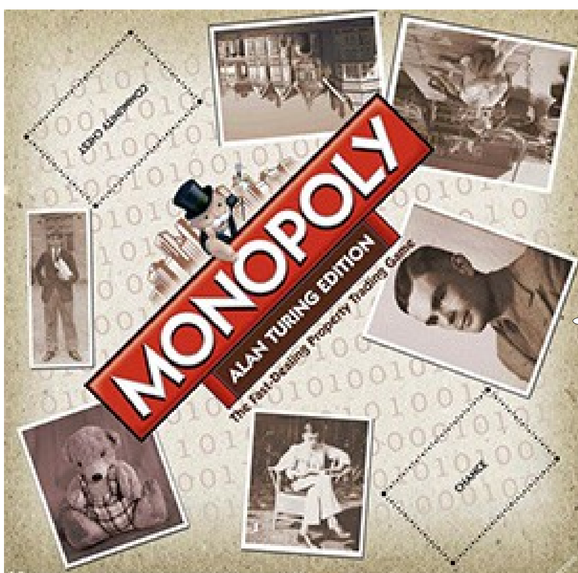
# Church & Turing

- The first response to the Entscheidungsproblem was by Alonzo Church (1935-6).
  - Using effective computability based on his λ-calculus.
  - Gave a negative answer!

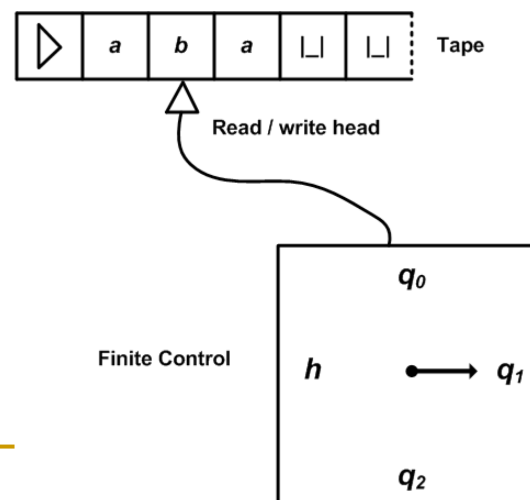
Church (1903-1995)

- Alan Turing (1936) postulated a simple, most general, mathematical model for computing – Turing machine (TM).

Turing (1912-1954)




100 years!



# Church & Turing

- Turing machines first appeared in the paper:

230 A. M. TURING [Nov. 12,

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. TURING.

[Received 28 May, 1936.—Read 12 November, 1936.]

[*Extracted from the Proceedings of the London Mathematical Society, Ser. 2, Vol. 42, 1937.*]

The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers

# Church & Turing


Gödel (1906-1978)

- Both the proofs were motivated by Kurt Gödel.
- Turing showed the undecidability of the Halting problem.
  - Deciding whether a given TM *halts* or not.
- Turing's proof idea for Entscheidungsproblem:
  - Enumerate the TMs as $\{M_1, M_2, M_3, \ldots\}$.
  - Let $M_i$ be the one solving the Halting problem.
  - Consider the TM $M$:

    On input x, if $M_i$ rejects x(x) then ACCEPT else NOT(x(x)).

  - What is $M(M)$ ?? ↯
  - Thus, Halting problem is undecidable. $\blacksquare$

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# Cook & Levin


Stephen Cook (1939-)

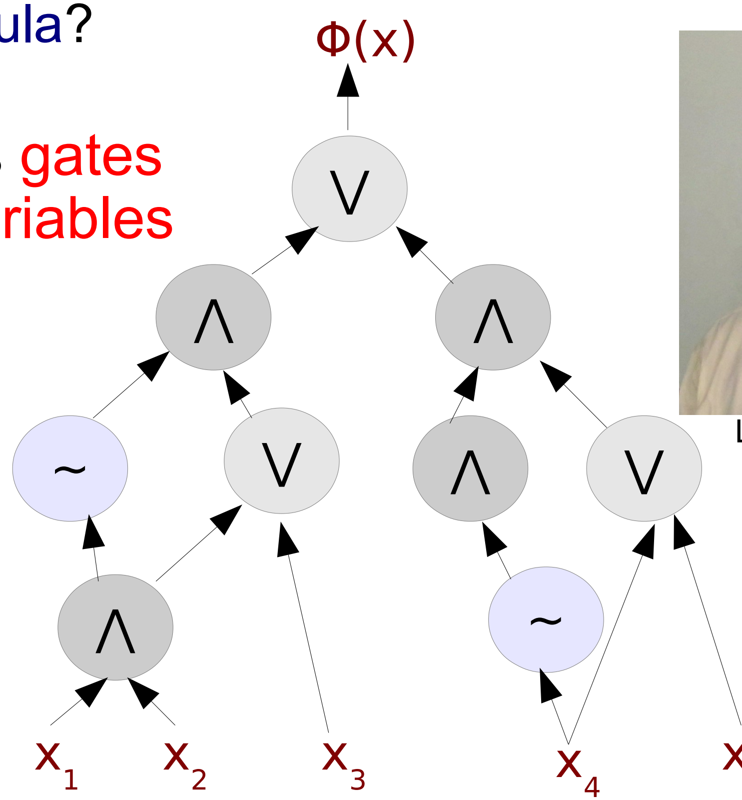
- Cook & Levin (1971) studied a more tractable version of the *Entscheidungsproblem*.
  - Truth of a boolean formula?


Leonid Levin (1948-)

- A boolean formula Φ has gates {AND, OR, NOT}, and variables $\{x_1, \ldots\ldots, x_n\}$ .

- We can try out all $2^n$ evaluations for truth.
  - Is there a faster way?

- Move from decidability to efficiency......



# Cook & Levin

- Intuitively, one cannot do any better than the *exponential*, i.e. $2^n$, time.
  - This is the $\mathcal{P}$ *vs* $\mathcal{NP}$ question.
  - Worth at least a *million* $$!

Clay Mathematics Institute (1999-)

- This is an extremely important problem because 100s of *practical* problems are known to be equivalent to it.
  - Karp (1972) himself showed 21 such problems!

Integer programming, set packing, vertex cover, feedback node set, hamiltonian cycle, chromatic number, clique, steiner tree, 3-dimensional matching, knapsack, job sequencing, partition, Max cut, independent set problem, Travelling salesmen problem

Richard Karp (1935-)

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# Valiant's permanent


Leslie Valiant (1949-)

- Valiant (1977) asked a related question.
  - Count the number of *good* evaluations of a given boolean formula.
  - Valiant's counting problem.
- Solving this would solve all our previous NP-hard problems.
- More interestingly, the counting problem reduces to a simple matrix question – Permanent.

$$Per\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}.$$

Given a matrix A compute Per(A) ?

# Valiant's permanent

- Notice that permanent looks very much like a determinant.

$$Det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}.$$

George Pólya (1887-1985)

- It is an old question of Pólya (1913): Can permanent be computed using the determinant?
- Valiant's study suggests that permanent is a much harder sibling of determinant!
- *Algebraic P vs NP question*:

  Is permanent efficiently computable ?

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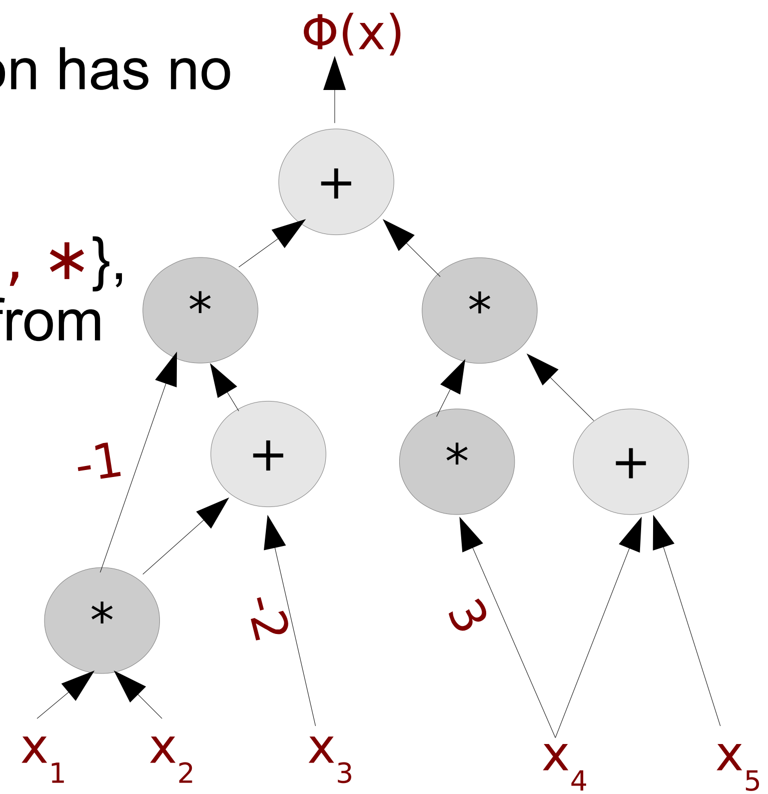
# Zero or nonzero

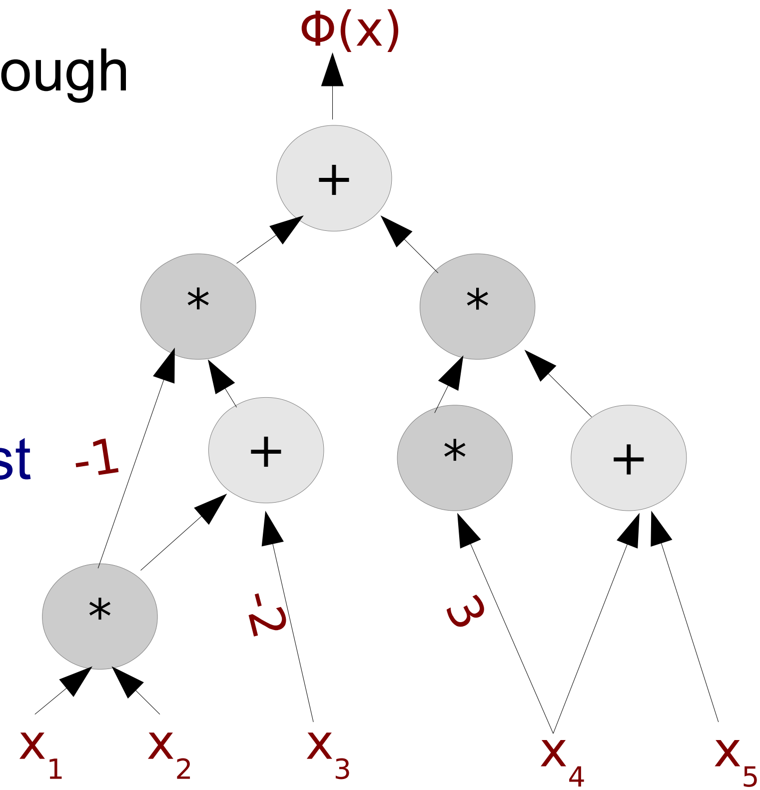
- The current research focuses on proving permanent's hardness – by algebraic means.
- I.e. show that the permanent function has no *small* arithmetic circuit.
- An arithmetic circuit $\Phi$ has gates $\{+, *\}$, variables $\{x_1,\ldots,x_n\}$ and constants from some field $F$.
- An arithmetic circuit is an algebraically neat model to capture real computation.



# Zero or nonzero

- *<u>Conjecture</u>: Permanent has no small arithmetic circuits.*
- Classical algebra is not developed enough to answer this question.
  - Permanent, circuits are both recent constructs.
  - A specialized theory is missing.
- As a warmup: Find an algorithm to test whether a given circuit is *zero*.
  - Identity testing.

*<u>Meta-Theorem</u>: A solution of identity testing would answer the permanent question.*



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# Fundamental Goals

- Find a *proper* algorithm for circuit *identity testing*.
- Prove *permanent* hardness against circuits.
- Resolve *algebraic* P vs NP (Valiant's *counting* problem).
- Resolve *P vs NP*.

*The tools therein shall enrich our understanding.*

*Thank you!*