Primality Testing, Randomness & Derandomization

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Bonn, May 2007

Outline

Primality: The Problem

- Given an integer n, test whether it is prime.
- Easy Solution: Divide n by all numbers between 2 and $(n-1)$.
- But we would like to do this in time polynomial in $\log n$.
- First asked by Kurt Gödel in a letter to John von Neumann in 1956.

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Primality: First Attempts

- Ancient Chinese (500 B.C.) and Greek (Eratosthenes, 200 B.C.) methods.
- Lucas proved in 1876 that $2^{127} - 1$ is prime; this would remain the largest known Mersenne prime for three-quarters of a century. Method generalizes to n with smooth $(n+1)$ (Lucas-Lehmer Test 1930s).
- Pépin's Test (1877): specialized for Fermat numbers $F_k = 2^{2^k} + 1$.
- Lucas Test (1891): When $(n-1)$ is smooth.
- Pocklington-Lehmer Test (1914): When we know distinct primes $p_1, \ldots, p_t | (n-1)$ such that $\prod_{i=1}^{t} p_t \geq \sqrt{n}$.

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Primality: Randomness Enters

Theorem (Solovay-Strassen, 1977)

An odd number n is prime iff for most $a \in \mathbb{Z}_n$, $a^{\frac{n-1}{2}} = \left(\frac{a}{n}\right)$.

- Jacobi symbol $\left(\frac{a}{n}\right)$ is computable in time $O^{\sim}(\log^2 n)$.
- We check the above equation for a random a.
- This gives a randomized test that takes time $O^{\sim}(\log^2 n)$.
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Randomized Algorithms

- These randomized primality tests began a vigorous study of randomness in computation.
- Deterministic Polynomial Time: P is the set of boolean functions $f : \{0,1\}^* \rightarrow \{0,1\}$ such that $f(x)$ is computable by a Turing machine in $|x|^c$ many steps.
- Randomized Polynomial Time: BPP is the set of boolean functions f such that:

$$\exists g \in \mathrm{P},\ \exists d > 0,\ \forall x,\ \Pr_{r \in \{0,1\}^{|x|^d}}[g(x,r) = f(x)] \geq \frac{2}{3}$$

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Randomized Algorithms: Examples

BPP contains some very well studied algebraic problems:

1. Primality testing.
2. Identity testing – given an arithmetic circuit $C(x_1, \ldots, x_n)$ computing a polynomial, test whether it is identically zero.
3. Polynomial factoring over finite fields – given a polynomial $f(x) \in \mathbb{F}_q[x]$, find a nontrivial factor.

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Determinism and Randomness: Different?

- Can we select the random bits carefully in a randomized algorithm such that there is no error?
- For example, if we assume GRH then the first $(2\log^2 n)$ a's suffice to test primality of n in Solovay-Strassen and Miller-Rabin tests.
- Can we derandomize any randomized polynomial time algorithm?
- Is BPP=P? or

"God does not play dice...."??

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Determinism and Randomness: Hardness Enters

- In the 1990s it was observed that if there are hard problems then they can be used to derandomize.
- Specifically, Impagliazzo-Wigderson showed in 1997 that BPP=P if E requires exponential boolean circuits.
- E is the set of boolean functions $f : \{0,1\}^* \rightarrow \{0,1\}$ such that $f(x)$ is computable by a Turing machine in $2^{c|x|}$ many steps.
- A boolean circuit comprises of AND, OR, NOT gates.

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- *Proof Idea:* Suppose $f \in$BPP and g is the corresponding boolean function computable in n^c time and using $m := n^d$ random bits.
- Suppose we have a hard boolean function $h \in$ E that cannot be computed in randomized polynomial time.
- Then instead of "feeding" g purely random m bits we can feed $h(y^{(1)}), \ldots, h(y^{(m)})$. Where $y^{(i)}$'s are substrings of a string $y \in \{0,1\}^{c \log n}$.
- We intend to show that these m bits would be random enough since h is hard.
- Thus, computing $g(x, h(y^{(1)}) \cdots h(y^{(m)}))$ for all $y \in \{0,1\}^{c \log n}$ and taking the majority vote would give us a deterministic polynomial time way to compute f.

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Proof that $h(y^{(1)}), \ldots, h(y^{(m)})$ *are pseudorandom:*

- Suppose $h(y^{(1)}), \ldots, h(y^{(m)})$ are not behaving randomly. Say:
  $$\left| \Pr_y \left[g(x, h(y^{(1)}) \cdots h(y^{(m)})) = f(x)\right] - \Pr_{r_1,\ldots,r_m} \left[g(x, r_1 \cdots r_m) = f(x)\right] \right| > \frac{1}{n}$$
  $$\Rightarrow \left| \sum_{i=0}^{m} \left(\Pr_{y,r_i,\ldots,r_m} \left[g(x, h(y^{(1)}) \cdots h(y^{(i-1)}) \cdot r_i \cdots r_m) = f(x)\right] - \Pr_{y,r_{i+1},\ldots,r_m} \left[g(x, h(y^{(1)}) \cdots h(y^{(i)}) \cdot r_{i+1} \cdots r_m) = f(x)\right]\right) \right| > \frac{1}{n}$$
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- Thus, g and f can be used to predict the value of $h(y^{(j)})$ which contradicts the hardness of h.

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  $$\Rightarrow | \sum_{i=0}^{m} \left(\Pr_{y,r_i,\ldots,r_m} \left[g(x, h(y^{(1)}) \cdots h(y^{(i-1)}) \cdot r_i \cdots r_m) = f(x)\right] - \Pr_{y,r_{i+1},\ldots,r_m} \left[g(x, h(y^{(1)}) \cdots h(y^{(i)}) \cdot r_{i+1} \cdots r_m) = f(x)\right]\right) | > \frac{1}{n}$$
- Therefore, $\exists j$ such that the prob of $g(x, h(y^{(1)}) \cdots h(y^{(j-1)}) \cdot r_j \cdots r_m) = f(x)$ differs from the prob of $g(x, h(y^{(1)}) \cdots h(y^{(j)}) \cdot r_{j+1} \cdots r_m) = f(x)$ by more than $\frac{1}{nm}$.
- Thus, g and f can be used to predict the value of $h(y^{(j)})$ which contradicts the hardness of h.

Derandomization

- We saw heuristic evidence that randomized polynomial time algorithms can always be made deterministic polynomial time.
- But such a general derandomization seems tied up with proving lower bounds.
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Outline

1. Prologue
2. Randomness
3. Derandomizing Primality Testing
4. Derandomizing Identity Testing
   - Depth 3 Circuits: Algorithm I
   - Depth 3 Circuits: Algorithm II
5. Epilogue

Agrawal-Kayal-S (AKS) Test

Theorem (A Generalization of FLT)

If n is a prime then for all $a \in \mathbb{Z}_n$, $(x+a)^n = (x^n + a) \pmod{n, x^r - 1}$.

- This was the basis of the AKS test proposed in 2002.
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AKS Test

1. If n is a prime power, it is composite.
2. Select an r such that $\mathrm{ord}_r(n) > 4\log^2 n$ and work in the ring $R := \mathbb{Z}_n[x]/(x^r - 1)$.
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- Suppose all the congruences hold and p is a prime factor of n.
- The group $I := \langle n, p \pmod{r} \rangle$. $t := \#I \geq \text{ord}_r(n) \geq 4\log^2 n$.
- The group $J := \langle (x+1), \ldots, (x+\ell) \pmod{p, h(x)} \rangle$ where $h(x)$ is an irreducible factor of $\frac{x^r-1}{x-1}$ modulo p.
  $\#J \geq 2^{\min\{t,\ell\}} > 2^{2\sqrt{t}\log n} \geq n^{2\sqrt{t}}$.
- *Proof:* Let $f(x), g(x)$ be two different products of $(x+a)$'s, having degree $< t$. Suppose $f(x) = g(x) \pmod{p, h(x)}$.
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Group $I := \langle n, p \pmod{r} \rangle$ is of size $t > 4\log^2 n$.
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AKS Test: Time Complexity

- Recall that r is the least number such that $\mathrm{ord}_r(n) > 4\log^2 n$.
- Prime number theorem gives $r = O(\log^5 n)$ and the algorithm takes time $O^{\sim}(\log^{10.5} n)$.
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Identities

- High School algebra teaches us lots of useful algebraic identities.
- For example,
  $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$.
- Lebesgue identity:

$$(a^2 + b^2 + c^2 + d^2)^2 = (a^2 + b^2 - c^2 - d^2)^2 + (2ac + 2bd)^2 + (2ad - 2bc)^2$$

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- Identity communicated by Euler in a letter to Goldbach on April 15, 1750:

$$
\begin{aligned}
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&\qquad (a_1b_1-a_2b_2-a_3b_3-a_4b_4)^2 \;+\; (a_1b_2+a_2b_1+a_3b_4-a_4b_3)^2 \;+\\
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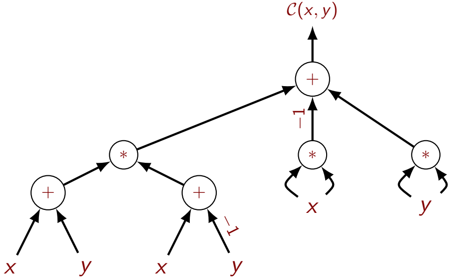

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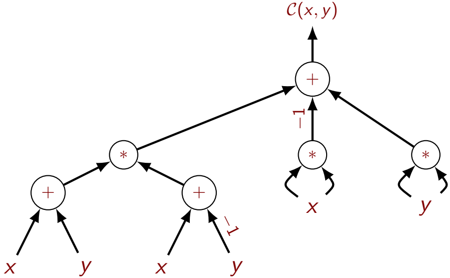

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Identity Testing is Important

Identity testing is instrumental in many complexity theory results:

- Graph matching is in RNC (Lovasz '79).
- PSPACE=IP (Shamir '90).
- NEXP=MIP (Babai-Fortnow-Lund '90).
- Even AKS Primality test is based on checking whether $(x+1)^n - (x^n + 1) = 0 \pmod{n}$ (Agrawal-Kayal-S '02).

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- (Schwartz '80, Zippel '79) gave a randomized algorithm for identity testing.
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The Question

Big question here: Can we do identity testing in deterministic polynomial time?

Consequences of Derandomization

- (Impagliazzo-Kabanets '03) showed that a derandomized identity test would imply circuit lower bounds for permanent.
- Thus, a derandomization of identity testing would both:
  - provide evidence that randomization in algorithms is dispensable, and
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- Some progress has been made for restricted circuits.
- Noncommutative formulas: (Raz-Shpilka '04) gave a deterministic polynomial time identity test.
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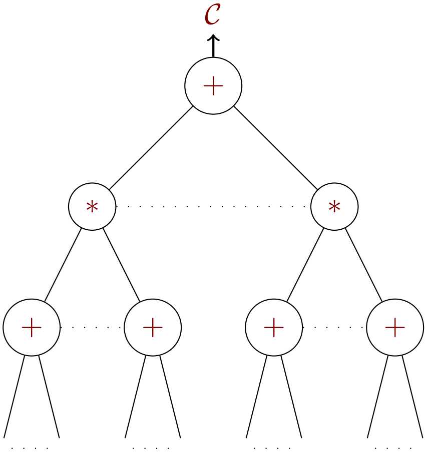

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be: $\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$
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- Pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
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- Note that in each recursive call:
  1. Fanin k reduces by atleast 1
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Idea for Diagonal Circuits

- Suppose the depth 3 circuit is a sum of powers of linear functions:

$$\mathcal{C}(x_1, \ldots, x_n) = \ell_1^d + \cdots + \ell_k^d$$

- The idea is to transform it into:
  $\sum_{i=1}^{k}(b_{i,1,0}+b_{i,1,1}x_1+\cdots+b_{i,1,d}x_1^d)\cdots(b_{i,n,0}+b_{i,n,1}x_n+\cdots+b_{i,n,d}x_n^d)$
- The above circuit can be viewed as a noncommutative circuit (*i.e.* $x_1, \ldots, x_n$ do not commutate).
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Idea for Diagonal Circuits: The Transformation

- We will present the circuit transformation over a field $\mathbb{F}$ of zero characteristic.
- Let $\mathcal{C}(x_1, \ldots, x_n) = \sum_{i=1}^{k} (a_{i,1}x_1 + \cdots + a_{i,n}x_n)^d$. Recall $\exp(x) = 1 + x + \frac{x^2}{2!} + \cdots$
- $\Rightarrow \quad (d!)^{-1} \cdot \mathcal{C}(x_1, \ldots, x_n) =$ coefficient of z^d in $\sum_{i=1}^{k} \exp\left((a_{1,1}x_1 + \cdots + a_{1,n}x_n)z\right)$.
- $=$ coefficient of z^d in $\sum_{i=1}^{k} \exp(a_{i,1}x_1 z) \cdots \exp(a_{i,n}x_n z)$.
- $=$ coefficient of z^d in $\sum_{i=1}^{k} \prod_{v=1}^{n} \left(1 + a_{i,v}x_v z + \cdots + \frac{a_{i,v}^d x_v^d z^d}{d!}\right)$.
- $= \sum_{i=1}^{k} \sum_{j=1}^{dn+1} z^{-dj} \cdot \prod_{v=1}^{n} \left(1 + a_{i,v}x_v z^j + \cdots + \frac{a_{i,v}^d x_v^d z^{dj}}{d!}\right) \pmod{z^{dn+1} - 1}$.

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- For concrete algebraic problems derandomization may be done if we understand the underlying structure well enough.
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