Demystifying the border of algebraic models

Joint works with Pranjal Dutta & Prateek Dwivedi. [CCC'21, FOCS'21, FOCS'22]

Nitin Saxena
CSE, IIT Kanpur

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TIFR, Mumbai

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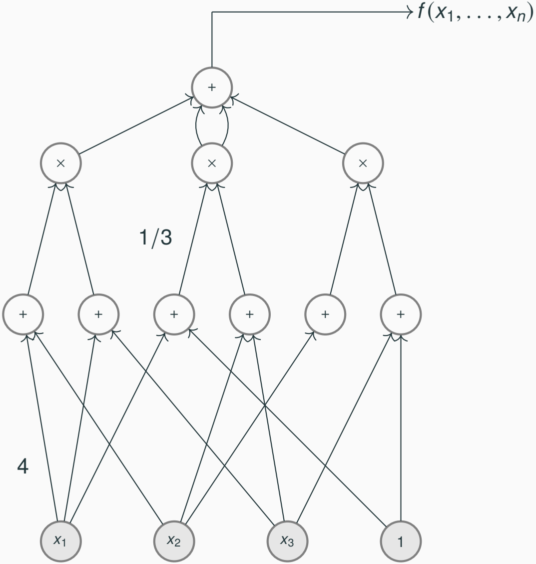
Basic Definitions and Terminologies

Algebraic circuits– VP



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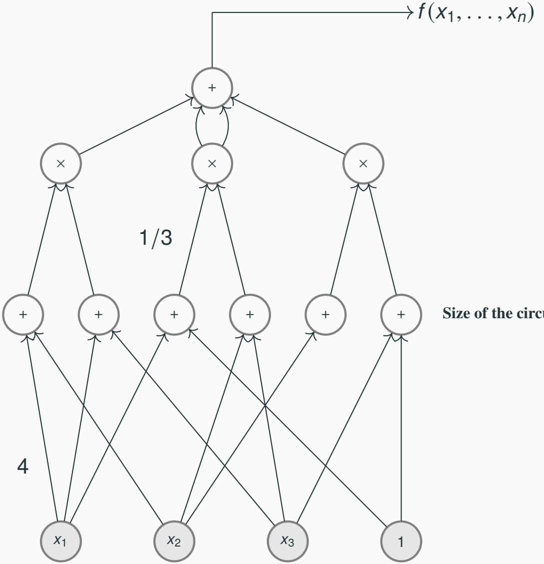
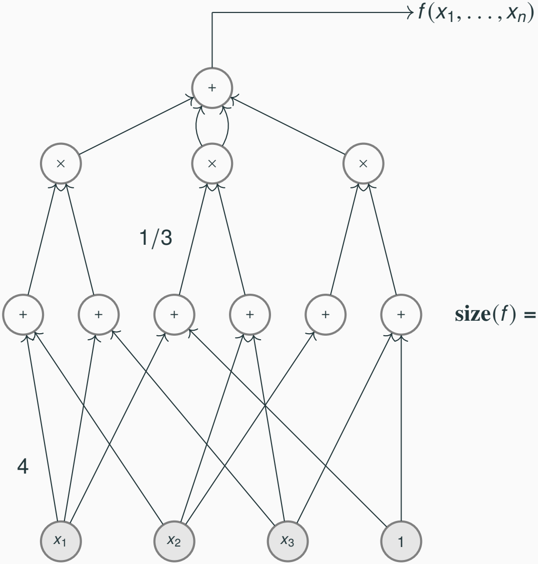
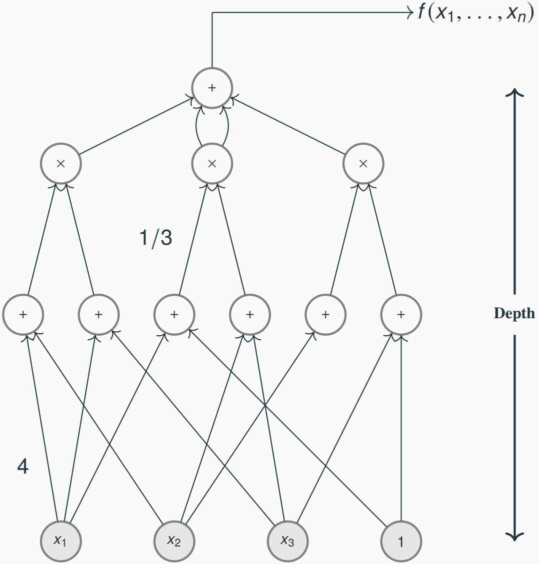


Size of the circuit = number of nodes + edges



size(f) **= min size of the circuit computing** f

Algebraic circuits– VP



The determinant polynomial– VBP

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- Let $X_s = [x_{i,j}]_{1 \le i,j \le s}$ be an $s \times s$ matrix of distinct variables $x_{i,j}$. Let $\mathrm{Sym}_s := \{\pi \mid \pi : \{1, \ldots, s\} \longrightarrow \{1, \ldots, s\} \text{ such that } \pi \text{ is bijective } \}$. Define

$$\mathsf{det}_s := \mathsf{det}(X_s) = \sum_{\pi \in \mathrm{Sym}_s} \mathrm{sgn}(\pi) \cdot \prod_{i=1}^{s} x_{i,\pi(i)} .$$

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- Relates tightly to Algebraic Branching Programs ABP, or IMM: Iterated Matrix Multiplication.

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- The minimum dimension of the matrix X_s to compute f, is called the **permanental complexity** $\mathsf{pc}(f)$.

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VNP = "explicit" (but "hard to compute"?) [Valiant 1979]

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Valiant's Conjecture [Valiant 1979]

$\mathsf{VBP} \neq \mathsf{VNP}$ & $\mathsf{VP} \neq \mathsf{VNP}$.

Equivalently, $\mathsf{dc}(\mathsf{perm}_n)$ and $\mathrm{size}(\mathsf{perm}_n)$ are both $n^{\omega(1)}$.

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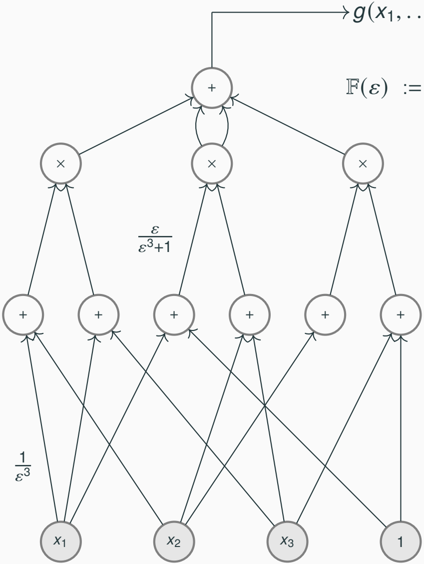
- This motivates a new model: *'approximative circuit'*.

$\rightarrow g(x_1, \ldots, x_n, \varepsilon) \in \mathbb{F}(\varepsilon)[\boldsymbol{x}]$



$\mathbb{F}(\varepsilon) := \left\{\frac{p(\varepsilon)}{q(\varepsilon)} \mid p, q \in \mathbb{F}[\varepsilon], q(\varepsilon) \neq 0\right\}$ is the fn.field

Algebraic approximation

- Suppose, we assume the following:
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- Bottomline: g_0 is **non-trivially** 'approximated' by the circuit, since $\lim_{\varepsilon \to 0} g(\boldsymbol{x}, \varepsilon) = g_0$.

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Algebraic Approximation [Bürgisser 2004]

A polynomial $h \in \mathbb{F}[\boldsymbol{x}]$ has approximative complexity s, if there is a circuit $g \in \mathbb{F}[\varepsilon][\boldsymbol{x}]$, of $\text{size}_{\mathbb{F}(\varepsilon)} = s$, and an error polynomial $S(\boldsymbol{x}, \varepsilon) \in \mathbb{F}[\varepsilon][\boldsymbol{x}]$ such that

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Border Depth-3 Circuits

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- How about $\overline{\Sigma^{[2]}\Pi\Sigma}$?

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2. Consider $A(\boldsymbol{x}) := \prod_{i=1}^{m}(1+\ell_i^d) = \prod_{i=1}^{m}\prod_{j=1}^{d}(\alpha_j+\ell_i)$, for $\alpha_j \in \mathbb{C}$.

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Proof of Kumar's result

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Proving Upper Bounds

De-bordering $\overline{\Sigma^{[2]}\Pi\Sigma}$ circuits

- If h is approximated by a $\Sigma^{[2]}\Pi\Sigma$ circuit with product fanin $\mathsf{poly}(n)$, what's the *exact* complexity of h?

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Remark. The result holds if one replaces the top-fanin-2 by arbitrary constant k.

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- ❑ We devise a technique called DiDIL - **Di**vide, **D**erive, **I**nduct with **L**imit.

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- $\mathsf{val}_z(\cdot)$ denotes the highest power of z dividing it (= least one across monomials). E.g., $h = \varepsilon z + \varepsilon^{-1} z^2 x_1 = (\varepsilon z) \cdot (1 + \varepsilon^{-2} z x_1)$. Then, $\mathsf{val}_z(h) = 1$.

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$$\begin{aligned} \Phi(f) + \varepsilon \cdot \Phi(S) &= \Phi(T_1) + \Phi(T_2) \\ \implies \Phi(f/T_2) + \varepsilon \cdot \Phi(S/T_2) &= \Phi(T_1/T_2) + 1 \\ \implies \partial_z \Phi(f/T_2) + \varepsilon \cdot \partial_z \Phi(S/T_2) &= \partial_z \Phi(T_1/T_2) =: g_1 \,. \end{aligned} \tag{1}$$

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- Both $\Phi(T_1)$ and $\Phi(T_2)$ have $\Pi\Sigma$ circuits (they have z and ε).

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- Suffices to compute $g_1 \bmod z^d$ and take the limit!

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- ❑ Note: *Definite* integration requires setting $z = 0$ in $\Phi(T_1/T_2) + 1$; that's why we need power-series in z.

$\square$

Proving Lower Bounds

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skip the section

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- Catch: But, $x_1 \cdot y_1 + \ldots + x_{k+1} \cdot y_{k+1} \in \overline{\Sigma^{[2]}\Pi^{O(k)}\Sigma}$!

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- Ambitious goal: Can we separate $\overline{\Sigma^{[k]}\Pi\Sigma}$ and $\overline{\Sigma^{[k+1]}\Pi\Sigma}$?
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- $x_1 \cdot y_1 + \ldots + x_{k+1} \cdot y_{k+1}$ cannot be computed by $\Sigma^{[k]}\Pi\Sigma$ circuits!
- Catch: But, $x_1 \cdot y_1 + \ldots + x_{k+1} \cdot y_{k+1} \in \overline{\Sigma^{[2]}\Pi^{O(k)}\Sigma}$!
- What lower bound works (if at all!)?

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- Classical is about *impossibility*. While, border is about *optimality*.

Proof Idea in $k = 2$: Non-homogeneity is all we need to care

- Three cases to consider:
  - Case I: T_1 and T_2 each has one linear polynomial $\ell_i \in \mathbb{F}(\varepsilon)[\boldsymbol{x}]$ as a factor, whose ε-free term is a linear form. Example: $\ell = (1 + \varepsilon)x_1 + \varepsilon x_2$,

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- So, **all-non-homogeneous** case is all that remains ...

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- $P_d(\boldsymbol{x}) + \varepsilon \cdot S(\boldsymbol{x}, \varepsilon) = T_1 + T_2$, where $T_i \in \Pi\Sigma \in \mathbb{F}(\varepsilon)[\boldsymbol{x}]$ have **all-non-homogeneous** linear factors.

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- Use the **least-monomial**, in z, to show that $P_d \in \overline{\Sigma^s \wedge \Sigma}$.
- Next, partial-derivative **measure**, in $\mathbf{x}$, implies $s \geq 2^{\Omega(d)}$!

$\square$

Conclusion

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Thank you! Questions?