Identities and Complexity

Nitin Saxena

Centrum voor Wiskunde en Informatica
Amsterdam

NVTI Theory Day 2007
Utrecht

Outline

Identities

- High School algebra teaches us lots of useful algebraic identities.
- For example, $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$.
- Lebesgue identity:

$$(a^2+b^2+c^2+d^2)^2 = (a^2+b^2-c^2-d^2)^2 + (2ac+2bd)^2 + (2ad-2bc)^2$$

Identities

- High School algebra teaches us lots of useful algebraic identities.
- For example, $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$.
- Lebesgue identity:

$$(a^2 + b^2 + c^2 + d^2)^2 = (a^2 + b^2 - c^2 - d^2)^2 + (2ac + 2bd)^2 + (2ad - 2bc)^2$$

Identities

- High School algebra teaches us lots of useful algebraic identities.
- For example, $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$.
- Lebesgue identity:

$$(a^2 + b^2 + c^2 + d^2)^2 = (a^2 + b^2 - c^2 - d^2)^2 + (2ac + 2bd)^2 + (2ad - 2bc)^2$$

Identities

- Identity communicated by Euler in a letter to Goldbach on April 15, 1750:

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) = (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4)^2 + (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)^2 + (a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2)^2 + (a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1)^2$$

- All these can be checked by expansion.

Identities

- Identity communicated by Euler in a letter to Goldbach on April 15, 1750:

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) = (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4)^2 + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)^2 + (a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)^2 + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)^2$$

- All these can be checked by expansion.

Bigger Identities

- Let p be an odd prime number. Then:

$$\sum_{i=1}^{p} \prod_{\substack{a_1,\ldots,a_m \in \mathbb{F}_p \\ a_1+\cdots+a_m = i \pmod{p}}} (y + a_1x_1 + \cdots + a_mx_m) = 0$$

- The polynomial on the LHS has degree: p^{m-1} .
- A naive expansion of the above produces exponentially many terms.
- Then how do we check the above identity efficiently ?

Bigger Identities

- Let p be an odd prime number. Then:

$$\sum_{i=1}^{p} \prod_{\substack{a_1,\ldots,a_m \in \mathbb{F}_p \\ a_1+\cdots+a_m = i \pmod{p}}} (y + a_1x_1 + \cdots + a_mx_m) = 0$$

- The polynomial on the LHS has degree: p^{m-1} .
- A naive expansion of the above produces exponentially many terms.
- Then how do we check the above identity efficiently ?

Bigger Identities

- Let p be an odd prime number. Then:

$$\sum_{i=1}^{p} \prod_{\substack{a_1,\dots,a_m \in \mathbb{F}_p \\ a_1+\cdots+a_m = i \pmod{p}}} (y + a_1x_1 + \cdots + a_mx_m) = 0$$

- The polynomial on the LHS has degree: p^{m-1} .
- A naive expansion of the above produces exponentially many terms.
- Then how do we check the above identity efficiently ?

Bigger Identities

- Let p be an odd prime number. Then:

$$\sum_{i=1}^{p} \prod_{\substack{a_1,\dots,a_m \in \mathbb{F}_p \\ a_1+\cdots+a_m = i \pmod{p}}} (y + a_1x_1 + \cdots + a_mx_m) = 0$$

- The polynomial on the LHS has degree: p^{m-1} .
- A naive expansion of the above produces exponentially many terms.
- Then how do we check the above identity efficiently ?

Randomness Helps

- Evaluate the above polynomial at a *random* point.
- It can be shown that "with high probability" the polynomial evaluates to zero iff it is an identity!
- But can this identity testing be done efficiently without using randomness?

Randomness Helps

- Evaluate the above polynomial at a *random* point.
- It can be shown that "with high probability" the polynomial evaluates to zero iff it is an identity!
- But can this identity testing be done efficiently without using randomness?

Randomness Helps

- Evaluate the above polynomial at a *random* point.
- It can be shown that "with high probability" the polynomial evaluates to zero iff it is an identity!
- But can this identity testing be done efficiently without using randomness?

Randomness Helps ??

- There are many problems with nice randomized efficient algorithms.
- Like, Identity testing, Primality testing, Polynomial factorization, Quicksort, Min-cut,......
- But there is a belief that randomness in polynomial time algorithms is always dispensable. In short:

"God does not play dice...."

Randomness Helps ??

- There are many problems with nice randomized efficient algorithms.
- Like, Identity testing, Primality testing, Polynomial factorization, Quicksort, Min-cut,......
- But there is a belief that randomness in polynomial time algorithms is always dispensable. In short:

  "God does not play dice...."

Randomness Helps ??

- There are many problems with nice randomized efficient algorithms.
- Like, Identity testing, Primality testing, Polynomial factorization, Quicksort, Min-cut,......
- But there is a belief that randomness in polynomial time algorithms is always dispensable. In short:

"God does not play dice...."

Randomness Helps ??

- There are many problems with nice randomized efficient algorithms.
- Like, Identity testing, Primality testing, Polynomial factorization, Quicksort, Min-cut,......
- But there is a belief that randomness in polynomial time algorithms is always dispensable. In short:

"God does not play dice...."

Randomness Helps ??

- Impagliazzo-Wigderson '96 showed that if there are "hard" functions in E then polynomial time randomized algorithms can be derandomized.
- Primality testing was successfully derandomized by Agrawal-Kayal-S in 2002.
- After Primality testing, arguably, the next most important problem waiting to be derandomized is identity testing.

Randomness Helps ??

- Impagliazzo-Wigderson '96 showed that if there are "hard" functions in E then polynomial time randomized algorithms can be derandomized.
- Primality testing was successfully derandomized by Agrawal-Kayal-S in 2002.
- After Primality testing, arguably, the next most important problem waiting to be derandomized is identity testing.

Randomness Helps ??

- Impagliazzo-Wigderson '96 showed that if there are "hard" functions in E then polynomial time randomized algorithms can be derandomized.
- Primality testing was successfully derandomized by Agrawal-Kayal-S in 2002.
- After Primality testing, arguably, the next most important problem waiting to be derandomized is identity testing.

Outline

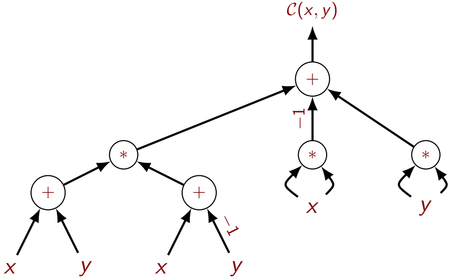
Formalizing Identity Testing

- We can assume that our polynomial expression is given in the form of an Arithmetic circuit $\mathcal{C}$:

- Identity testing is the problem of checking whether a given circuit is zero or not.

Formalizing Identity Testing

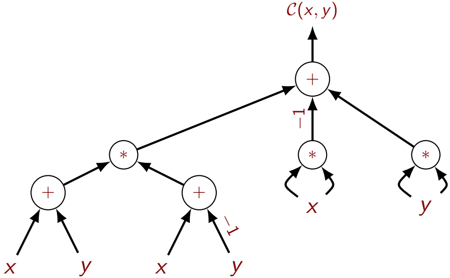
- We can assume that our polynomial expression is given in the form of an Arithmetic circuit $\mathcal{C}$:



- Identity testing is the problem of checking whether a given circuit is zero or not.

Formalizing Identity Testing

- We can assume that our polynomial expression is given in the form of an Arithmetic circuit $\mathcal{C}$:



- Identity testing is the problem of checking whether a given circuit is zero or not.

A Randomized Solution

- Schwartz '80, Zippel '79 gave a randomized algorithm for identity testing.
- Given an arithmetic circuit $C(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$:
  - Pick a random tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
  - Return YES iff $C(\alpha_1, \ldots, \alpha_n) = 0$.
- Clearly, this can be done in time polynomial in the size of C.

A Randomized Solution

- Schwartz '80, Zippel '79 gave a randomized algorithm for identity testing.
- Given an arithmetic circuit $C(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$:
  - Pick a random tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
  - Return YES iff $C(\alpha_1, \ldots, \alpha_n) = 0$.
- Clearly, this can be done in time polynomial in the size of C.

A Randomized Solution

- Schwartz '80, Zippel '79 gave a randomized algorithm for identity testing.
- Given an arithmetic circuit $C(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$:
  - Pick a random tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
  - Return YES iff $C(\alpha_1, \ldots, \alpha_n) = 0$.
- Clearly, this can be done in time polynomial in the size of C.

A Randomized Solution

- Schwartz '80, Zippel '79 gave a randomized algorithm for identity testing.
- Given an arithmetic circuit $C(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$:
  - Pick a random tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
  - Return YES iff $C(\alpha_1, \ldots, \alpha_n) = 0$.
- Clearly, this can be done in time polynomial in the size of C.

A Randomized Solution

- Schwartz '80, Zippel '79 gave a randomized algorithm for identity testing.
- Given an arithmetic circuit $C(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$:
  - Pick a random tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
  - Return YES iff $C(\alpha_1, \ldots, \alpha_n) = 0$.
- Clearly, this can be done in time polynomial in the size of C.

A Randomized Solution

Proof of Correctness:

- If C is a zero circuit then clearly the algorithm returns YES for any choice of $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
- Say, $C(x_1, \ldots, x_n)$ is computing a nonzero polynomial of total degree d.
- It can be shown that:

$$\text{Prob}_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{F}^n}\left[C(\alpha_1, \ldots, \alpha_n) = 0\right] \leq \frac{d}{\#\mathbb{F}}$$

- Thus, for a suitably large $\mathbb{F}$, $\frac{d}{\#\mathbb{F}} \leq \frac{1}{2}$.
- Thus, with a good chance we will pick a non-root of C.

A Randomized Solution

Proof of Correctness:

- If C is a zero circuit then clearly the algorithm returns YES for any choice of $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
- Say, $C(x_1, \ldots, x_n)$ is computing a nonzero polynomial of total degree d.
- It can be shown that:

$$\mathrm{Prob}_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{F}^n}\left[C(\alpha_1,\ldots,\alpha_n)=0\right] \leq \frac{d}{\#\mathbb{F}}$$

- Thus, for a suitably large $\mathbb{F}$, $\frac{d}{\#\mathbb{F}} \leq \frac{1}{2}$.
- Thus, with a good chance we will pick a non-root of C.

A Randomized Solution

Proof of Correctness:

- If C is a zero circuit then clearly the algorithm returns YES for any choice of $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
- Say, $C(x_1, \ldots, x_n)$ is computing a nonzero polynomial of total degree d.
- It can be shown that:

$$\text{Prob}_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{F}^n}\left[C(\alpha_1, \ldots, \alpha_n) = 0\right] \leq \frac{d}{\#\mathbb{F}}$$

- Thus, for a suitably large $\mathbb{F}$, $\frac{d}{\#\mathbb{F}} \leq \frac{1}{2}$.
- Thus, with a good chance we will pick a non-root of C.

A Randomized Solution

Proof of Correctness:

- If C is a zero circuit then clearly the algorithm returns YES for any choice of $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
- Say, $C(x_1, \ldots, x_n)$ is computing a nonzero polynomial of total degree d.
- It can be shown that:

$$\text{Prob}_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{F}^n}\left[C(\alpha_1,\ldots,\alpha_n) = 0\right] \leq \frac{d}{\#\mathbb{F}}$$

- Thus, for a suitably large $\mathbb{F}$, $\frac{d}{\#\mathbb{F}} \leq \frac{1}{2}$.
- Thus, with a good chance we will pick a non-root of C.

A Randomized Solution

Proof of Correctness:

- If C is a zero circuit then clearly the algorithm returns YES for any choice of $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$.
- Say, $C(x_1, \ldots, x_n)$ is computing a nonzero polynomial of total degree d.
- It can be shown that:

$$\mathrm{Prob}_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{F}^n}\left[C(\alpha_1, \ldots, \alpha_n) = 0\right] \leq \frac{d}{\#\mathbb{F}}$$

- Thus, for a suitably large $\mathbb{F}$, $\frac{d}{\#\mathbb{F}} \leq \frac{1}{2}$.
- Thus, with a good chance we will pick a non-root of C.

The Question

Big question here: Can we do identity testing in deterministic polynomial time?

Connections

Identity testing is instrumental in many complexity theory results:

- Graph matching problems have efficient randomized parallel algorithms (Lovasz '79).
- PSPACE has interactive protocols (Shamir '90).
- NEXP has two-prover interactive protocols (Babai-Fortnow-Lund '90).
- The first deterministic polynomial time Primality test was based on checking whether $(x+1)^n - (x^n + 1) = 0 \pmod{n}$ (Agrawal-Kayal-S '02).

Connections

Identity testing is instrumental in many complexity theory results:

- Graph matching problems have efficient randomized parallel algorithms (Lovasz '79).
- PSPACE has interactive protocols (Shamir '90).
- NEXP has two-prover interactive protocols (Babai-Fortnow-Lund '90).
- The first deterministic polynomial time Primality test was based on checking whether $(x+1)^n - (x^n+1) = 0 \pmod{n}$ (Agrawal-Kayal-S '02).

Connections

Identity testing is instrumental in many complexity theory results:

- Graph matching problems have efficient randomized parallel algorithms (Lovasz '79).
- PSPACE has interactive protocols (Shamir '90).
- NEXP has two-prover interactive protocols (Babai-Fortnow-Lund '90).
- The first deterministic polynomial time Primality test was based on checking whether $(x+1)^n - (x^n + 1) = 0 \pmod{n}$ (Agrawal-Kayal-S '02).

Connections

Identity testing is instrumental in many complexity theory results:

- Graph matching problems have efficient randomized parallel algorithms (Lovasz '79).
- PSPACE has interactive protocols (Shamir '90).
- NEXP has two-prover interactive protocols (Babai-Fortnow-Lund '90).
- The first deterministic polynomial time Primality test was based on checking whether $(x+1)^n - (x^n + 1) = 0 \pmod{n}$ (Agrawal-Kayal-S '02).

Deeper Connections

- (Impagliazzo-Kabanets '03) showed that a derandomized identity test would imply circuit lower bounds for NEXP.
- Thus, a derandomization of identity testing would both:
  - provide evidence that randomization in algorithms is dispensable, and
  - give circuit lower bounds.

Deeper Connections

- (Impagliazzo-Kabanets '03) showed that a derandomized identity test would imply circuit lower bounds for NEXP.
- Thus, a derandomization of identity testing would both:
  - provide evidence that randomization in algorithms is dispensable, and
  - give circuit lower bounds.

Deeper Connections

- (Impagliazzo-Kabanets '03) showed that a derandomized identity test would imply circuit lower bounds for NEXP.
- Thus, a derandomization of identity testing would both:
    - provide evidence that randomization in algorithms is dispensable, and
    - give circuit lower bounds.

Deeper Connections

- (Impagliazzo-Kabanets '03) showed that a derandomized identity test would imply circuit lower bounds for NEXP.
- Thus, a derandomization of identity testing would both:
  - provide evidence that randomization in algorithms is dispensable, and
  - give circuit lower bounds.

Outline

Progress

- Some progress has been made when the input circuit has bounded many levels.
- Multilinear circuits of depth 3: (Raz-Shpilka '04) gave a deterministic polynomial time identity test.
- Circuits of depth 3 with bounded top fanin: (Kayal-S '06) gave a deterministic polynomial time identity test.

Progress

- Some progress has been made when the input circuit has bounded many levels.
- Multilinear circuits of depth 3: (Raz-Shpilka '04) gave a deterministic polynomial time identity test.
- Circuits of depth 3 with bounded top fanin: (Kayal-S '06) gave a deterministic polynomial time identity test.

Progress

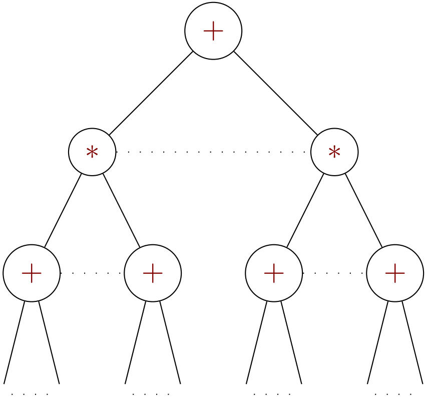
- Some progress has been made when the input circuit has bounded many levels.
- Multilinear circuits of depth 3: (Raz-Shpilka '04) gave a deterministic polynomial time identity test.
- Circuits of depth 3 with bounded top fanin: (Kayal-S '06) gave a deterministic polynomial time identity test.

Depth 3 Circuits: The Setting

- For identity testing, it is sufficient to consider a "sum of product of linear functions" (ΣΠΣ *circuit*).

Depth 3 Circuits: The Setting

- For identity testing, it is sufficient to consider a "sum of product of linear functions" ($\Sigma\Pi\Sigma$ *circuit*).



Depth 3 Circuits: The Setting

- Our input circuit $\mathcal{C}$ over a field $\mathbb{F}$ will look like:

$$\mathcal{C}(z_1, \ldots, z_n) = T_1 + \cdots + T_k$$

where T_i is a product of linear functions $L_{i,1}, \ldots, L_{i,d}$

where $L_{i,j} = (a_{i,j,0} + a_{i,j,1}z_1 + \cdots + a_{i,j,n}z_n)$, a's $\in \mathbb{F}$.

Depth 3 Circuits: The Setting

- Our input circuit $\mathcal{C}$ over a field $\mathbb{F}$ will look like:

$$\mathcal{C}(z_1, \ldots, z_n) = T_1 + \cdots + T_k$$

where T_i is a product of linear functions $L_{i,1}, \ldots, L_{i,d}$
where $L_{i,j} = (a_{i,j,0} + a_{i,j,1}z_1 + \cdots + a_{i,j,n}z_n)$, a's $\in \mathbb{F}$.

Depth 3 Circuits: The Setting

- Our input circuit $\mathcal{C}$ over a field $\mathbb{F}$ will look like:

$$\mathcal{C}(z_1, \ldots, z_n) = T_1 + \cdots + T_k$$

where T_i is a product of linear functions $L_{i,1}, \ldots, L_{i,d}$

where $L_{i,j} = (a_{i,j,0} + a_{i,j,1}z_1 + \cdots + a_{i,j,n}z_n)$, a's $\in \mathbb{F}$.

Depth 3 Circuits: The Setting

- Our input circuit $\mathcal{C}$ over a field $\mathbb{F}$ will look like:

$$\mathcal{C}(z_1, \ldots, z_n) = T_1 + \cdots + T_k$$

where T_i is a product of linear functions $L_{i,1}, \ldots, L_{i,d}$
where $L_{i,j} = (a_{i,j,0} + a_{i,j,1}z_1 + \cdots + a_{i,j,n}z_n)$, a's $\in \mathbb{F}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$
$$\text{where } T_i = L_{i,1} \cdots L_{i,d}$$

- Pick $(d+1)$ coprime linear functions $p_1, \ldots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \ldots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \pmod{x_1}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$

where $T_i = L_{i,1} \cdots L_{i,d}$

- Pick $(d+1)$ coprime linear functions $p_1, \ldots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \ldots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \pmod{x_1}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$
$$\text{where } T_i = L_{i,1} \cdots L_{i,d}$$

- Pick $(d+1)$ coprime linear functions $p_1, \ldots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \ldots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \pmod{x_1}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$
$$\text{where } T_i = L_{i,1} \cdots L_{i,d}$$

- Pick $(d+1)$ coprime linear functions $p_1, \ldots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \ldots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \pmod{x_1}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \dots, x_n) = T_1 + \cdots + T_k$$
$$\text{where } T_i = L_{i,1} \cdots L_{i,d}$$

- Pick $(d+1)$ coprime linear functions $p_1, \dots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \dots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \dots, \tau(x_n)) = 0 \pmod{x_1}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$
$$\text{where } T_i = L_{i,1} \cdots L_{i,d}$$

- Pick $(d+1)$ coprime linear functions $p_1, \ldots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \ldots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \pmod{x_1}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$
$$\text{where } T_i = L_{i,1} \cdots L_{i,d}$$

- Pick $(d+1)$ coprime linear functions $p_1, \ldots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \ldots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \pmod{x_1}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$
$$\text{where } T_i = L_{i,1} \cdots L_{i,d}$$

- Pick $(d+1)$ coprime linear functions $p_1, \ldots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \ldots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \pmod{x_1}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$
$$\text{where } T_i = L_{i,1} \cdots L_{i,d}$$

- Pick $(d+1)$ coprime linear functions $p_1, \ldots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \ldots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \pmod{x_1}$.

The Idea of Chinese Remaindering

- Let $\mathcal{C}$ be:

$$\mathcal{C}(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$
$$\text{where } T_i = L_{i,1} \cdots L_{i,d}$$

- Pick $(d+1)$ coprime linear functions $p_1, \ldots, p_{d+1}$ from the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- $\mathcal{C} = 0$ **iff** for all $i \in [d+1]$, $\mathcal{C} = 0 \pmod{p_i}$.
- $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i}$ can be checked recursively because:
  - $\mathcal{C}$ modulo p_i has top fanin atmost $(k-1)$ because for some j, $T_j = 0 \pmod{p_i}$.
  - Let τ be an invertible map on $x_1, \ldots, x_n$ sending $p_i \mapsto x_1$.
  - Then $\mathcal{C} = 0 \pmod{p_i}$ **iff** $\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \pmod{x_1}$.

Chinese Remaindering needs generalization

- There may not always be $(d+1)$ coprime linear functions in the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- So we need to pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
  1. every $p_i^{e_i}$ divides some T_j.
  2. $e_1 + \cdots + e_\ell \geq d$.
- How do we check $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i^{e_i}}$?
- We transform $p_i \mapsto x_1$ by applying an invertible map τ on $x_1, \ldots, x_n$. Then $\mathcal{C} = 0 \pmod{p_i^{e_i}}$ **iff**

$$\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \text{ over } \mathbb{F}[x_1]/(x_1^{e_i}).$$

- Thus, we recursively solve identity testing over "bigger" rings.

Skip details

Chinese Remaindering needs generalization

- There may not always be $(d+1)$ coprime linear functions in the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- So we need to pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
  1. every $p_i^{e_i}$ divides some T_j.
  2. $e_1 + \cdots + e_\ell \geq d$.
- How do we check $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i^{e_i}}$?
- We transform $p_i \mapsto x_1$ by applying an invertible map τ on $x_1, \ldots, x_n$. Then $\mathcal{C} = 0 \pmod{p_i^{e_i}}$ iff

$$\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \text{ over } \mathbb{F}[x_1]/(x_1^{e_i}).$$

- Thus, we recursively solve identity testing over "bigger" rings.

Skip details

Chinese Remaindering needs generalization

- There may not always be $(d+1)$ coprime linear functions in the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- So we need to pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
  1. every $p_i^{e_i}$ divides some T_j.
  2. $e_1 + \cdots + e_\ell \geq d$.
- How do we check $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i^{e_i}}$?
- We transform $p_i \mapsto x_1$ by applying an invertible map τ on $x_1, \ldots, x_n$. Then $\mathcal{C} = 0 \pmod{p_i^{e_i}}$ **iff**

$$\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \text{ over } \mathbb{F}[x_1]/(x_1^{e_i}).$$

- Thus, we recursively solve identity testing over “bigger” rings.

Skip details

Chinese Remaindering needs generalization

- There may not always be $(d+1)$ coprime linear functions in the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- So we need to pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
  1. every $p_i^{e_i}$ divides some T_j.
  2. $e_1 + \cdots + e_\ell \geq d$.
- How do we check $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i^{e_i}}$?
- We transform $p_i \mapsto x_1$ by applying an invertible map τ on $x_1, \ldots, x_n$. Then $\mathcal{C} = 0 \pmod{p_i^{e_i}}$ iff

$$\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \text{ over } \mathbb{F}[x_1]/(x_1^{e_i}).$$

- Thus, we recursively solve identity testing over "bigger" rings.

Skip details

Chinese Remaindering needs generalization

- There may not always be $(d+1)$ coprime linear functions in the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- So we need to pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
  1. every $p_i^{e_i}$ divides some T_j.
  2. $e_1 + \cdots + e_\ell \ \geq \ d$.
- How do we check $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i^{e_i}}$?
- We transform $p_i \mapsto x_1$ by applying an invertible map τ on $x_1, \ldots, x_n$. Then $\mathcal{C} = 0 \pmod{p_i^{e_i}}$ iff

$$\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \text{ over } \mathbb{F}[x_1]/(x_1^{e_i}).$$

- Thus, we recursively solve identity testing over "bigger" rings.

Skip details

Chinese Remaindering needs generalization

- There may not always be $(d+1)$ coprime linear functions in the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- So we need to pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
  1. every $p_i^{e_i}$ divides some T_j.
  2. $e_1 + \cdots + e_\ell \geq d$.
- How do we check $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i^{e_i}}$?
- We transform $p_i \mapsto x_1$ by applying an invertible map τ on $x_1, \ldots, x_n$. Then $\mathcal{C} = 0 \pmod{p_i^{e_i}}$ **iff**

$$\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \text{ over } \mathbb{F}[x_1]/(x_1^{e_i}).$$

- Thus, we recursively solve identity testing over “bigger” rings.

Skip details

Chinese Remaindering needs generalization

- There may not always be $(d+1)$ coprime linear functions in the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- So we need to pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
  1. every $p_i^{e_i}$ divides some T_j.
  2. $e_1 + \cdots + e_\ell \;\geq\; d$.
- How do we check $\mathcal{C} \overset{?}{=} 0 \pmod{p_i^{e_i}}$?
- We transform $p_i \mapsto x_1$ by applying an invertible map τ on $x_1, \ldots, x_n$. Then $\mathcal{C} = 0 \pmod{p_i^{e_i}}$ **iff**

$$\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \text{ over } \mathbb{F}[x_1]/(x_1^{e_i}).$$

- Thus, we recursively solve identity testing over “bigger” rings.

Skip details

Chinese Remaindering needs generalization

- There may not always be $(d+1)$ coprime linear functions in the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- So we need to pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
  1. every $p_i^{e_i}$ divides some T_j.
  2. $e_1 + \cdots + e_\ell \geq d$.
- How do we check $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i^{e_i}}$?
- We transform $p_i \mapsto x_1$ by applying an invertible map τ on $x_1, \ldots, x_n$. Then $\mathcal{C} = 0 \pmod{p_i^{e_i}}$ **iff**

$$\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \text{ over } \mathbb{F}[x_1]/(x_1^{e_i}).$$

- Thus, we recursively solve identity testing over “bigger” rings.

Skip details

Chinese Remaindering needs generalization

- There may not always be $(d+1)$ coprime linear functions in the set $\{L_{i,j} \mid i \in [k], j \in [d]\}$.
- So we need to pick powers $p_1^{e_1}, \ldots, p_\ell^{e_\ell}$ of coprime linear functions $p_1, \ldots, p_\ell$ such that,
  1. every $p_i^{e_i}$ divides some T_j.
  2. $e_1 + \cdots + e_\ell \geq d$.
- How do we check $\mathcal{C} \stackrel{?}{=} 0 \pmod{p_i^{e_i}}$?
- We transform $p_i \mapsto x_1$ by applying an invertible map τ on $x_1, \ldots, x_n$. Then $\mathcal{C} = 0 \pmod{p_i^{e_i}}$ **iff**

$$\mathcal{C}(\tau(x_1), \ldots, \tau(x_n)) = 0 \text{ over } \mathbb{F}[x_1]/(x_1^{e_i}).$$

- Thus, we recursively solve identity testing over "bigger" rings.

Skip details

Identity Test (in more detail)

- Let R be a local subring of $\mathbb{F}[x_1, \ldots, x_m]$ with maximal ideal $\mathcal{M}$.
- Let the input be a $\Sigma\Pi\Sigma$ circuit $\mathcal{C}(z_1, \ldots, z_n)$ in $R[z_1, \ldots, z_n]$:
  $\mathcal{C} = T_1 + \cdots + T_k$
  where, $T_i = L_{i,1} \cdots L_{i,d}$
- Wlog let T_1 produce the lexicographically largest monomial.
- T_1 can be factored into *coprime* polynomials as follows:
  $T_1 = \alpha \cdot p_1(z_1, \ldots, z_n) \cdots p_s(z_1, \ldots, z_n)$
  where, $p_i = (\ell_i + m_{i,1}) \cdots (\ell_i + m_{i,d_i})$ for some linear form ℓ_i and α, $m_{i,j}$'s are in $\mathcal{M}$.

Identity Test (in more detail)

- Let R be a local subring of $\mathbb{F}[x_1, \ldots, x_m]$ with maximal ideal $\mathcal{M}$.
- Let the input be a $\Sigma\Pi\Sigma$ circuit $\mathcal{C}(z_1, \ldots, z_n)$ in $R[z_1, \ldots, z_n]$:
  $\mathcal{C} = T_1 + \cdots + T_k$
  where, $T_i = L_{i,1} \cdots L_{i,d}$
- Wlog let T_1 produce the lexicographically largest monomial.
- T_1 can be factored into *coprime* polynomials as follows:
  $T_1 = \alpha \cdot p_1(z_1, \ldots, z_n) \cdots p_s(z_1, \ldots, z_n)$
  where, $p_i = (\ell_i + m_{i,1}) \cdots (\ell_i + m_{i,d_i})$ for some linear form ℓ_i and α, $m_{i,j}$'s are in $\mathcal{M}$.

Identity Test (in more detail)

- Let R be a local subring of $\mathbb{F}[x_1, \ldots, x_m]$ with maximal ideal $\mathcal{M}$.
- Let the input be a $\Sigma\Pi\Sigma$ circuit $\mathcal{C}(z_1, \ldots, z_n)$ in $R[z_1, \ldots, z_n]$:
  $\mathcal{C} = T_1 + \cdots + T_k$
  where, $T_i = L_{i,1} \cdots L_{i,d}$
- Wlog let T_1 produce the lexicographically largest monomial.
- T_1 can be factored into *coprime* polynomials as follows:
  $T_1 = \alpha \cdot p_1(z_1, \ldots, z_n) \cdots p_s(z_1, \ldots, z_n)$
  where, $p_i = (\ell_i + m_{i,1}) \cdots (\ell_i + m_{i,d_i})$ for some linear form ℓ_i and α, $m_{i,j}$'s are in $\mathcal{M}$.

Identity Test (in more detail)

- Let R be a local subring of $\mathbb{F}[x_1, \ldots, x_m]$ with maximal ideal $\mathcal{M}$.
- Let the input be a $\Sigma\Pi\Sigma$ circuit $\mathcal{C}(z_1, \ldots, z_n)$ in $R[z_1, \ldots, z_n]$:
  $\mathcal{C} = T_1 + \cdots + T_k$
  where, $T_i = L_{i,1} \cdots L_{i,d}$
- Wlog let T_1 produce the lexicographically largest monomial.
- T_1 can be factored into *coprime* polynomials as follows:
  $T_1 = \alpha \cdot p_1(z_1, \ldots, z_n) \cdots p_s(z_1, \ldots, z_n)$
  where, $p_i = (\ell_i + m_{i,1}) \cdots (\ell_i + m_{i,d_i})$ for some linear form ℓ_i and α, $m_{i,j}$'s are in $\mathcal{M}$.

Identity Test (in more detail)

- Let R be a local subring of $\mathbb{F}[x_1,\ldots,x_m]$ with maximal ideal $\mathcal{M}$.
- Let the input be a $\Sigma\Pi\Sigma$ circuit $\mathcal{C}(z_1,\ldots,z_n)$ in $R[z_1,\ldots,z_n]$:
  $\mathcal{C} = T_1 + \cdots + T_k$
  where, $T_i = L_{i,1}\cdots L_{i,d}$
- Wlog let T_1 produce the lexicographically largest monomial.
- T_1 can be factored into *coprime* polynomials as follows:
  $T_1 = \alpha \cdot p_1(z_1,\ldots,z_n)\cdots p_s(z_1,\ldots,z_n)$
  where, $p_i = (\ell_i + m_{i,1})\cdots(\ell_i + m_{i,d_i})$ for some linear form ℓ_i
  and $\alpha,\ m_{i,j}$'s are in $\mathcal{M}$.

Identity Test (in more detail)

- Let R be a local subring of $\mathbb{F}[x_1, \ldots, x_m]$ with maximal ideal $\mathcal{M}$.
- Let the input be a $\Sigma\Pi\Sigma$ circuit $\mathcal{C}(z_1, \ldots, z_n)$ in $R[z_1, \ldots, z_n]$:
  $\mathcal{C} = T_1 + \cdots + T_k$
  where, $T_i = L_{i,1} \cdots L_{i,d}$
- Wlog let T_1 produce the lexicographically largest monomial.
- T_1 can be factored into *coprime* polynomials as follows:
  $T_1 = \alpha \cdot p_1(z_1, \ldots, z_n) \cdots p_s(z_1, \ldots, z_n)$
  where, $p_i = (\ell_i + m_{i,1}) \cdots (\ell_i + m_{i,d_i})$ for some linear form ℓ_i and α, $m_{i,j}$'s are in $\mathcal{M}$.

Identity Test (in more detail)

- Let R be a local subring of $\mathbb{F}[x_1, \ldots, x_m]$ with maximal ideal $\mathcal{M}$.
- Let the input be a $\Sigma\Pi\Sigma$ circuit $\mathcal{C}(z_1, \ldots, z_n)$ in $R[z_1, \ldots, z_n]$:
  $\mathcal{C} = T_1 + \cdots + T_k$
  where, $T_i = L_{i,1} \cdots L_{i,d}$
- Wlog let T_1 produce the lexicographically largest monomial.
- T_1 can be factored into *coprime* polynomials as follows:
  $T_1 = \alpha \cdot p_1(z_1, \ldots, z_n) \cdots p_s(z_1, \ldots, z_n)$
  where, $p_i = (\ell_i + m_{i,1}) \cdots (\ell_i + m_{i,d_i})$ for some linear form ℓ_i
  and α, $m_{i,j}$'s are in $\mathcal{M}$.

Identity Test (in more detail)

- Let R be a local subring of $\mathbb{F}[x_1, \ldots, x_m]$ with maximal ideal $\mathcal{M}$.
- Let the input be a $\Sigma\Pi\Sigma$ circuit $\mathcal{C}(z_1, \ldots, z_n)$ in $R[z_1, \ldots, z_n]$:
  $\mathcal{C} = T_1 + \cdots + T_k$
  where, $T_i = L_{i,1} \cdots L_{i,d}$
- Wlog let T_1 produce the lexicographically largest monomial.
- T_1 can be factored into *coprime* polynomials as follows:
  $T_1 = \alpha \cdot p_1(z_1, \ldots, z_n) \cdots p_s(z_1, \ldots, z_n)$
  where, $p_i = (\ell_i + m_{i,1}) \cdots (\ell_i + m_{i,d_i})$ for some linear form ℓ_i and $\alpha,\ m_{i,j}$'s are in $\mathcal{M}$.

Identity Test (in more detail)

- $\mathcal{C}(z_1, \ldots, z_n) = 0$ **iff**

  for all $i \in [s]$, $\mathcal{C} = 0 \pmod{p_i}$

  and

  lexicographically largest monomial of $\mathcal{C}$ has zero coefficient.

- For a fixed i: transform $\ell_i \mapsto z_1$ by an invertible linear transformation τ_i on $z_1, \ldots, z_n$ and, thus, $p_i \mapsto (z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})$
- Then $\mathcal{C} = 0 \pmod{p_i}$ **iff**

$$\tau_i(\mathcal{C}) = 0 \pmod{(z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})}.$$

- This entails checking $\tau_i(T_2) + \cdots + \tau_i(T_k) = 0$ over the local ring $R[z_1]/\left((z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})\right)$.
- Thus, we can recursively check whether $\mathcal{C} = 0 \pmod{p_i}$.

Identity Test (in more detail)

- $\mathcal{C}(z_1, \ldots, z_n) = 0$ **iff**

  for all $i \in [s]$, $\mathcal{C} = 0 \pmod{p_i}$

  and

  lexicographically largest monomial of $\mathcal{C}$ has zero coefficient.

- For a fixed i: transform $\ell_i \mapsto z_1$ by an invertible linear transformation τ_i on $z_1, \ldots, z_n$ and, thus, $p_i \mapsto (z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})$

- Then $\mathcal{C} = 0 \pmod{p_i}$ **iff**

$$\tau_i(\mathcal{C}) = 0 \pmod{(z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})}.$$

- This entails checking $\tau_i(T_2) + \cdots + \tau_i(T_k) = 0$ over the local ring $R[z_1]/((z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i}))$.

- Thus, we can recursively check whether $\mathcal{C} = 0 \pmod{p_i}$.

Identity Test (in more detail)

- $\mathcal{C}(z_1, \ldots, z_n) = 0$ **iff**

  for all $i \in [s]$, $\mathcal{C} = 0 \pmod{p_i}$

  and

  lexicographically largest monomial of $\mathcal{C}$ has zero coefficient.

- For a fixed i: transform $\ell_i \mapsto z_1$ by an invertible linear transformation τ_i on $z_1, \ldots, z_n$ and, thus, $p_i \mapsto (z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})$
- Then $\mathcal{C} = 0 \pmod{p_i}$ **iff**

$$\tau_i(\mathcal{C}) = 0 \pmod{(z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})}.$$

- This entails checking $\tau_i(T_2) + \cdots + \tau_i(T_k) = 0$ over the local ring $R[z_1]/\left((z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})\right)$.
- Thus, we can recursively check whether $\mathcal{C} = 0 \pmod{p_i}$.

Identity Test (in more detail)

- $\mathcal{C}(z_1, \ldots, z_n) = 0$ **iff**

  $$\text{for all } i \in [s],\ \mathcal{C} = 0 \pmod{p_i}$$
  and
  lexicographically largest monomial of $\mathcal{C}$ has zero coefficient.
- For a fixed i: transform $\ell_i \mapsto z_1$ by an invertible linear transformation τ_i on $z_1, \ldots, z_n$ and, thus, $p_i \mapsto (z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})$
- Then $\mathcal{C} = 0 \pmod{p_i}$ **iff**

  $$\tau_i(\mathcal{C}) = 0 \pmod{(z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})}.$$
- This entails checking $\tau_i(T_2) + \cdots + \tau_i(T_k) = 0$ over the local ring $R[z_1]/((z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i}))$.
- Thus, we can recursively check whether $\mathcal{C} = 0 \pmod{p_i}$.

Identity Test (in more detail)

- $\mathcal{C}(z_1, \ldots, z_n) = 0$ **iff**

  for all $i \in [s]$, $\mathcal{C} = 0 \pmod{p_i}$

  and

  lexicographically largest monomial of $\mathcal{C}$ has zero coefficient.
- For a fixed i: transform $\ell_i \mapsto z_1$ by an invertible linear transformation τ_i on $z_1, \ldots, z_n$ and, thus, $p_i \mapsto (z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})$
- Then $\mathcal{C} = 0 \pmod{p_i}$ **iff**

$$\tau_i(\mathcal{C}) = 0 \pmod{(z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})}.$$

- This entails checking $\tau_i(T_2) + \cdots + \tau_i(T_k) = 0$ over the local ring $R[z_1]/\left((z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})\right)$.
- Thus, we can recursively check whether $\mathcal{C} = 0 \pmod{p_i}$.

Identity Test (in more detail)

- $\mathcal{C}(z_1, \ldots, z_n) = 0$ **iff**

  for all $i \in [s]$, $\mathcal{C} = 0 \pmod{p_i}$

  and

  lexicographically largest monomial of $\mathcal{C}$ has zero coefficient.
- For a fixed i: transform $\ell_i \mapsto z_1$ by an invertible linear transformation τ_i on $z_1, \ldots, z_n$ and, thus, $p_i \mapsto (z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})$
- Then $\mathcal{C} = 0 \pmod{p_i}$ **iff**

$$\tau_i(\mathcal{C}) = 0 \pmod{(z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})}.$$

- This entails checking $\tau_i(T_2) + \cdots + \tau_i(T_k) = 0$ over the local ring $R[z_1]/\left((z_1 + m_{i,1}) \cdots (z_1 + m_{i,d_i})\right)$.
- Thus, we can recursively check whether $\mathcal{C} = 0 \pmod{p_i}$.

Time Complexity

- Note that in each recursive call:
  1. Fanin k reduces by atleast 1
  2. Dimension of the base ring increases atmost d times.
- The computations that we do are on rings of dimension atmost d^k.
- Identity testing for depth 3 circuits over n variables, total degree d and top fanin k can be done in time $poly(d^k, n)$.

Time Complexity

- Note that in each recursive call:
    1. Fanin k reduces by atleast 1
    2. Dimension of the base ring increases atmost d times.
- The computations that we do are on rings of dimension atmost d^k.
- Identity testing for depth 3 circuits over n variables, total degree d and top fanin k can be done in time $poly(d^k, n)$.

Time Complexity

- Note that in each recursive call:
  1. Fanin k reduces by atleast 1
  2. Dimension of the base ring increases atmost d times.
- The computations that we do are on rings of dimension atmost d^k.
- Identity testing for depth 3 circuits over n variables, total degree d and top fanin k can be done in time $poly(d^k, n)$.

Time Complexity

- Note that in each recursive call:
  1. Fanin k reduces by atleast 1
  2. Dimension of the base ring increases atmost d times.
- The computations that we do are on rings of dimension atmost d^k.
- Identity testing for depth 3 circuits over n variables, total degree d and top fanin k can be done in time $poly(d^k, n)$.

Time Complexity

- Note that in each recursive call:
  1. Fanin k reduces by atleast 1
  2. Dimension of the base ring increases atmost d times.
- The computations that we do are on rings of dimension atmost d^k.
- Identity testing for depth 3 circuits over n variables, total degree d and top fanin k can be done in time $poly(d^k, n)$.

Outline

In Conclusion

- Depth 3 Identity testing for bounded top fanin is in P.
- Open Problem: Identity testing for general depth 3 circuits ?
- "Easier" Open Problem: Identity testing for a *diagonalized* $\Sigma\Pi\Sigma$ circuit, i.e.,

$$C(x_1, \ldots, x_n) = L_1^d + \cdots + L_k^d$$

where, $L_1, \ldots, L_k$ are linear functions.

Questions?

In Conclusion

- Depth 3 Identity testing for bounded top fanin is in P.
- Open Problem: Identity testing for general depth 3 circuits ?
- "Easier" Open Problem: Identity testing for a *diagonalized* $\Sigma\Pi\Sigma$ circuit, i.e.,

$$C(x_1, \ldots, x_n) = L_1^d + \cdots + L_k^d$$

where, $L_1, \ldots, L_k$ are linear functions.

Questions?

In Conclusion

- Depth 3 Identity testing for bounded top fanin is in P.
- Open Problem: Identity testing for general depth 3 circuits ?
- "Easier" Open Problem: Identity testing for a *diagonalized* $\Sigma\Pi\Sigma$ circuit, i.e.,

$$C(x_1, \ldots, x_n) = L_1^d + \cdots + L_k^d$$

where, $L_1, \ldots, L_k$ are linear functions.

Questions?

In Conclusion

- Depth 3 Identity testing for bounded top fanin is in P.
- Open Problem: Identity testing for general depth 3 circuits ?
- "Easier" Open Problem: Identity testing for a *diagonalized* $\Sigma\Pi\Sigma$ circuit, i.e.,

$$C(x_1, \ldots, x_n) = L_1^d + \cdots + L_k^d$$

where, $L_1, \ldots, L_k$ are linear functions.

Questions?

In Conclusion

- Depth 3 Identity testing for bounded top fanin is in P.
- Open Problem: Identity testing for general depth 3 circuits ?
- "Easier" Open Problem: Identity testing for a *diagonalized* $\Sigma\Pi\Sigma$ circuit, i.e.,

$$C(x_1, \ldots, x_n) = L_1^d + \cdots + L_k^d$$

where, $L_1, \ldots, L_k$ are linear functions.

Questions?

In Conclusion

- Depth 3 Identity testing for bounded top fanin is in P.
- Open Problem: Identity testing for general depth 3 circuits ?
- "Easier" Open Problem: Identity testing for a *diagonalized* $\Sigma\Pi\Sigma$ circuit, i.e.,

$$C(x_1, \ldots, x_n) = L_1^d + \cdots + L_k^d$$

where, $L_1, \ldots, L_k$ are linear functions.

Questions?