A largish sum-of-squares implies circuit hardness (& derandomization)

Nitin Saxena (CSE@IIT Kanpur, India)

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Contents

- Sum-of-Squares (SOS) Representation
- SOS hardness

- Algebraic Circuits
- SOS hardness => Circuit hardness

- Blackbox Identity Testing (PIT)

- Sum-of-Cubes (SOC) hardness
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Sum-of-Squares (SOS) Representation

- For a polynomial f over $\mathbb{F}$, the SOS representation is:
  - $f = c_1 \cdot f_1^2 + \ldots + c_t \cdot f_t^2$, where $c_i \in \mathbb{F}$, $f_i \in \mathbb{F}[x_1, \ldots, x_n]$.
  - *Size* is number of monomials $\sum_i |f_i|_0$.
  - Denote the minimal size by support-sum $S(f)$.
- It's a *complete* model, if $\mathrm{char}\ \mathbb{F} \neq 2$.
  - Trivially, $S(f) \leq 4 \cdot |f|_0$.
- For simplicity, consider *univariate* SOS representations ($n=1$).
- **Example**: For deg$<d$ univariate $f(x)$, simply use monomials
  $$\{ x^i , x^{i\sqrt{d}} \mid 0 \leq i < \sqrt{d} \} .$$
  - (Agrawal'20) $t = 2 \cdot \sqrt{d}$ many squares suffice for any f.
  - Overall, expect $S(f) \geq 2\sqrt{d} \cdot 2\sqrt{d} = 4d$.

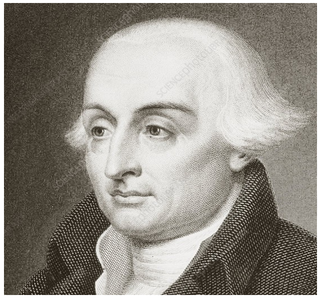
SOS Representation

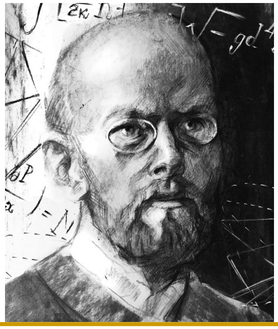
- Does there exist degree-d $f(x)$ with $S(f) \geq \Omega(d)$?
  - By *dimension*-argument it exists!
  - Assume $\mathbb{F} = \mathbb{C}$.

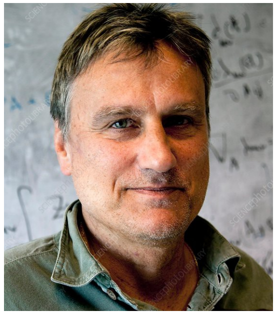
- To be of any help in complexity theory, we have to study SOS for polynomials that are explicit.
  - We would work with several definitions.
  - Eg. $(x+1)^d$ is `explicit'.

SOS Representation – History

- (1770) Lagrange's 4-squares thm: *Integer* as SOS of 4 squares.
  - Several such examples in number theory (Ramanujan 1917).
  - Pythagorean triples, Fermat's 2-squares, Legendre's 3-squares
- (1900) Hilbert's 17th Problem: *Positive Real polynomials* as SOS of rational functions?
  - Note: $c_i = 1$.
- (1990s) SOS constraints in convex optimization.
  - *Lasserre hierarchy* of relaxations in SDP (based on deg).







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SOS Hardness

- Defn: A degree-d $f(x)$ is explicit if it's *coefficient-function* coef(x^i)(f) is `easy':
  - Given (i,j) the j-th bit of coef(x^i)(f) is *polylog(d)-time*.
  - Or, ...is in $\#P/poly$.
  - Or, ...is in CH.

Sub-constant/ vanishing fn?

- **SOS-hard:** There's an *explicit* f and $\varepsilon>0$ with $S(f)> d^{\varepsilon+0.5}$.
  - $\varepsilon=0$ trivial. Existentially, much stronger property holds.

- There are numerous candidates for $f(x)$:
  - $(x+1)^d$
  - $\sum_i 2^{i^2} x^i$
  - $\prod_i (x+i)$

extremely small circuit complexity!

Yet useful?

$\mathbf{exp(x)}_{\leq d} := \sum^d_{i=0} x^i/i!$

$\sum_i 2^i x^i$ is not a candidate!

SOS Hardness – Comparisons

- Concept is quite weak/ incomparable to earlier ones about uni/multi-variate polynomials. As they needed sum-of unbounded-powers (or `power'*ful*):
  - (AV'08)..(GKKS'13)..(AGS'18) *Hardness* for special depth-4/3.
  - (Koiran'10) *Tau-conjecture* about roots of depth-4 expressions.
  - (KPTT'15) *Newton-polygon-Tau-conjecture* for sum-of unbounded-powers.
  - (Raz'08) *Super-poly-elusive* functions eluding degree-2 maps.

- $(x+1)^d$ good candidate for SOS-hardness. Not so, for the earlier conjectures.

- SOS-hard (n-variate): There's *explicit* $f(x_1,\ldots,x_n)$ and $\varepsilon>0$ with $S(f) > \{n+d \text{ choose } n\}^{\varepsilon+0.5}$.
  - Constant n.

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Algebraic Circuits

- Circuit has addition/multiplication *gates*; connected by *wires*.
  - Input variables at *leaves* are $x_1,\dots,x_n$; output $f(\bar{x})$.
  - size(f) is minimum graph-size of such a circuit.
- (1979) Valiant's Conjecture: $VP \neq VNP$.
  - VP – polynomial-families, poly(n)-degree, poly(n)-size.
  - VNP – exp.sum over a VP polynomial-family.

Achieved for *constant-depth* circuits! [LST21]

- Reduces to *highly-specialized* depth-4,3/width-2 questions.
  - ...(VSBR'83)...(AV'08)(R'08)(R'10)...(SSS'09)...(K'11)...(GKKS'13)...(KPTT'15)(KKPS'15)...(AGS'18)...
  - ***Qn:*** *Does it reduce to a model as weak as SOS(1-var)?*
- **Goal:** Squash circuit to SOS(n-var) with *nontrivial* property.
  - Else, it won't lift to proving circuit lower bounds.
  - **Hint:** Few squares, Low-degrees.

Algebraic Circuits – to SOS(n-var)

- (VSBR'83) $\deg(f) \le d$, $\text{size}(f) \le s$ can be rewritten:
  - Exists circuit C' of size $\text{poly}(sd)$ and depth $\log d$.
  - Exists formula F of size $s^{O(\log d)}$ and depth $\log d$.
  - Exists ABP B of size $s^{O(\log d)}$; *layers-d homogeneous*.

- Cut at the $d/2$ layer to get:
  - $f = \sum_{i \le |B|} f_{i,1} f_{i,2}$, where $\deg(f_{i,j}) \le d/2$.

- Use $4f_1 f_2 = (f_1+f_2)^2 - (f_1-f_2)^2$ to derive:

- **Theorem.1:** $\deg(f) \le d$, $\text{size}(f) \le s$ implies $f = \sum_{i \le s'} f_i^2$
  - where $s' \le s^{O(\log d)}$ and $\deg(f_i) \le d/2$.

❑

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- **Theorem.2:** SOS-hard implies VP ≠ VNP.

- *Pf idea:* Consider SOS-hard $f(x)$. Define $(k-1)^{\varepsilon} \geq 6$. Convert f to *multilinear*, kn-*variate*, *degree*-n polynomial $F(\bar{y})$.
  - Monomial x^i in $f(x)$ maps to $\varphi(x^i) :=$
  - $\Pi\{ y_{j,l} \mid l \cdot k^{j-1}$ contributes *place-value* in $base_k(i)$ }.
  - $k^n \geq d+1 > (k-1)^n$. So, $n := \Theta(\varepsilon \cdot \log d)$. F is kn-variate.
  - Suppose $size(F) \leq d^{\mu}$. Thm.1 gives SOS s.t.
  - $S(F) \leq (d^{\mu} n)^{O(\log n)} \cdot \{kn + n/2 \text{ choose } n/2\}$
  - $\leq d^{O(\mu \log n)} \cdot (6(k-1))^{n/2}$
  - $\leq d^{o(\varepsilon)} \cdot (k-1)^{(1+\varepsilon)n/2} \leq d^{o(\varepsilon)+(1+\varepsilon)/2} < d^{0.5+\varepsilon}$.
  - $S(f) \leq S(\varphi f) = S(F) \leq d^{\varepsilon+0.5}$ contradicts SOS-hardness.
  - Thus, $F \in VNP$ & $> d^{\mu} = (kn)^{\omega(1)}$ hard.
  - Finally, $F \in VNP \setminus VP$. ❑

$\mu := (\log d \cdot \log\log d)^{-0.5} > \omega(1/ \varepsilon \cdot \log d)$

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Blackbox poly.id.testing (PIT)

- Given circuit $C(x_1, \ldots, x_n)$ of size s, whether it is zero?
  - In $\mathrm{poly}(s)$ many bit operations?
  - Only $\mathbb{F}$ = finite field, rationals.
  - Brute-force expansion is as expensive as s^s.

- Randomization gives a practical, *blackbox* solution.
  - Evaluate $C(x_1, \ldots, x_n)$ at a random point in $\mathbb{F}^n$. [P.I.Lemma]
  - (Ore 1922), (DeMillo & Lipton 1978), (Zippel 1979), (Schwartz 1980).

- Blackbox PIT is equivalent to designing hitting-set $H \subset \mathbb{F}^n$.
  - H contains non-root of *each* $C(x_1, \ldots, x_n)$ of size s.

- Appears in many CS contexts (both algos/lower bounds):
  - ...(Lovász'79)(Heintz,Schnorr'79)(Blum,et.al'80)(Babai,et.al'90)(Clausen,et.al'91)(AKS'02)(KI'04)(A'05,'06)(Klivans, Shpilka'06)(DSY'09)(SV'10)(Mulmuley'11,'12,'17)(Kopparty,Saraf, Shpilka'14)(Pandey,S,Sinhababu'16)(Guo,S, Sinhababu'18)....<many more>

Blackbox poly.id.testing (PIT)

- Deterministic PIT algos known *only for restricted* models.
  - Too diverse to list here...

SUBEXP PIT for *constant-depth* circuits! [LST21]

- PIT exhibits some *amazing* phenomena:
  - *Specific* hitting-sets => VP ≠ VNP. (A'11)(K'11,KP'11).
  - Hitting-sets *strongly* bootstrap. (AGS'18)(KST'19)(GKSS'19)
  - Exp.hardness => Hitting-sets in QuasiP ($s^{O(\log s)}$). (KI'04)
  - Recall …reduces to *highly-specialized* depth-4,3/width-2.
- **Qn:** Could SOS-hardness imply complete PIT?
  - Up to QuasiP implied by Thm.2.
  - Issue with *older conjectures* that imply VP ≠ VNP.

Give only log-var reduction, not O(1)-var

- *We don't know....* [Thm.2/1 are `weak': #Vars? Deg in SOS?]
  - Modify Thm.2/1's proof to connect SOC (sum-of-cubes).

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Sum-of-Cubes (SOC) Hardness

- For a polynomial f over $\mathbb{F}$, the SOC representation is:
  - $f = c_1 \cdot f_1^3 + \ldots + c_t \cdot f_t^3$, where $c_i \in \mathbb{F}$, $f_i \in \mathbb{F}[x_1,\ldots,x_n]$.
  - *Support-union* is *distinct* monomials $\cup_i \mathrm{supp}(f_i)$.
  - Denote the minimal size by support-union $U(f,t)$.
- **SOC-hard:** There's poly(d)-time-*explicit* f and constant $\varepsilon' < 1/2$ with $U(f, d^{\varepsilon'}) \geq \Omega(d)$.
  - Seems false over $\mathbb{F} = \mathbb{C}, \mathbb{R}$. [dim.argument]
  - Instead fix $\mathbb{F} = \mathbb{Q}$ – natural choice for PIT.
  - *(Agrawal'20: False, if $\varepsilon' \geq 1/2$.)*

$x^2+y^2=3$: $\mathbb{R}$-roots; but no $\mathbb{Q}$-root.

- Again, numerous candidates for $f(x)$:
  - $(x+1)^d$, $\sum_i 2^{i^2} x^i$, $\prod_i (x+i)$, ….

$\mathbf{exp(x)}_{\leq d} := \sum_{i=0}^{d} x^i/i!$

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SOC Hardness => Blackbox PIT

- **Theorem.3:** SOC-hard implies blackbox-PIT in P.

- *Pf idea:* Consider SOC-hard $f(x)$: $U(f,d^{\varepsilon'}) \geq \delta\cdot d$. Convert f to k-*variate*, *ind-degree*-n polynomial $F(\bar{y})$.
  - Monomial x^i in $f(x)$ maps to $\varphi(x^i) :=$
  - $\Pi\{ y_j^{\,l} \mid l\cdot(n+1)^{j-1}$ contributes *place-value* in $\mathrm{base}_{n+1}(i) \}$.
  - $(n+1)^k \geq d+1 > n^k$. So, $n := O(d^{1/k})$. F is k-variate.
  - Let $\mathrm{size}(F) \leq d^{\mu}$. Thm.1(SOC), gives $(d^{\mu}\cdot kn)^c$ cubes of 4/11-th degree :
  - $U(F, d^{(\mu+1/k)c}) \leq \{k + 4kn/11$ choose $k\}$
  - $\leq (e+ 4e\cdot n/11)^k < n^k \cdot (10.9/11)^k \leq \delta\cdot d$.
  - Contradicts $U(f,d^{\varepsilon'}) \geq \delta\cdot d$.
  - => F is $k=O(1)$-*variate, ideg-n,* poly(n^k)-time-*explicit,* and
  - hardness $d^{\mu} \geq n^{\mu k} > \deg(F)^3$.
  - Apply (GKSS'19) for complete PIT. ❑

Ensure $\varepsilon'/c > (\mu+1/k)$,
$\mu\cdot k \geq 4$

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At the end …

- Largish SOS strong enough for circuit lower bounds.
  - $\deg(f_i)$'s restricted below $\tilde{O}(d)$.

- SOS falls a *bit short* of derandomization. But, SOC suffices.
  - Could we improve this part?

- Qn: Is SOC-hardness *heuristically* true (over $\mathbb{F} = \mathbb{Q}$) ?
  - Hybrid-Qn for SOS: $\varepsilon' < 1/2 < \varepsilon$ with $U(f, d^{\varepsilon'}) > d^{\varepsilon}$?
  - **=>** Thm.2 works as well!

- Prove: there's *sub*-constant ε with $S((x+1)^d) > d^{\varepsilon+0.5}$, over $\mathbb{F} = \mathbb{C}$.

> $\sqrt{d}.(\log d)$?

Thank you!