Prime Numbers and Circuits

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23rd European Conference on Operational Research
Bonn, 8th July 2009

1. Brief History of Primes
2. Primality testing
3. Derandomization?
4. Circuits
5. Primality Derandomized
6. Questions

Outline

Prime Numbers

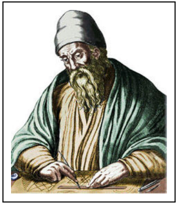

Fig: Euclid

- An integer $n > 1$ is *prime* if its divisors are only 1 and n.
- They are the building blocks of numbers and this means, as Euclid demonstrated in 300 B.C., primes are infinitely many.
- Not only are they pervasive in Mathematics but also appear in practice eg. Cryptography, Communication, ...
- So how do we check and find primes?

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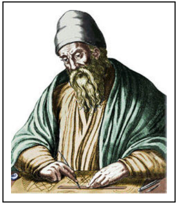

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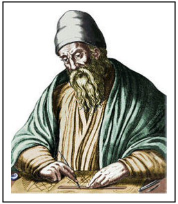

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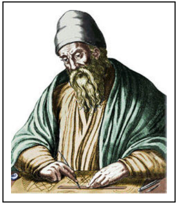

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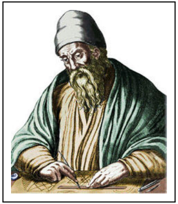

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Eratosthenes & his sieve

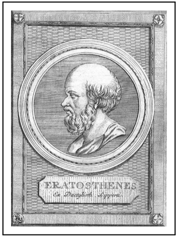

Fig: Eratosthenes

Fig: The Sieve

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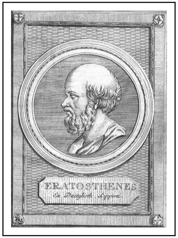

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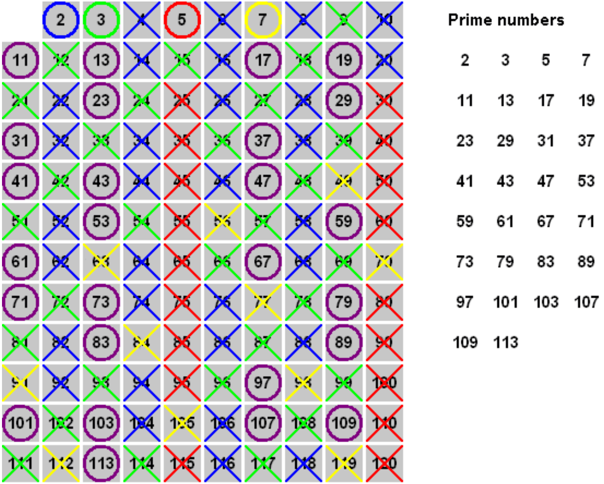

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Eratosthenes Sieve

"Sift the Twos and sift the Threes, The Sieve of Eratosthenes. When the multiples sublime, The numbers that remain are Prime."

- This is the high school method to test primes, attributed to Eratosthenes 200 B.C.
- For a number n, it is sufficient to divide by numbers upto $\sqrt{n}$.
- Thus, it takes around $O(\sqrt{n})$ steps. For a 100-bit number this means 2^{50} steps!

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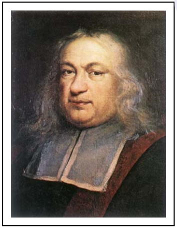
Theorem (Fermat, 1660s)

If n is prime then for every a, $a^n = a \pmod{n}$.



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- Can we ascertain the primality of n by checking $a^n = a \pmod{n}$ for few *magical* a?
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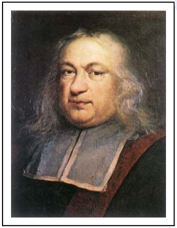

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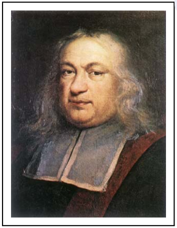

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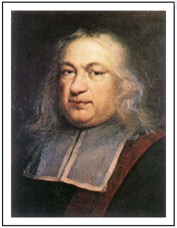

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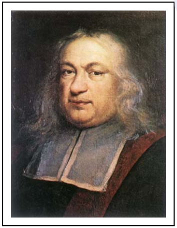

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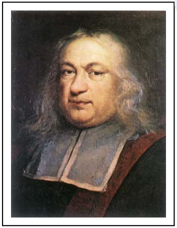

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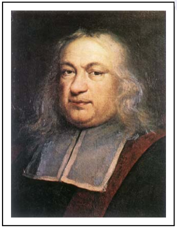

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Prime Number Estimates

Fig: Gauss

- For any real $x > 1$, let $\pi(x)$ be the number of primes $p \leq x$.
- By looking at the tables of primes Legendre and Gauss (independently) conjectured in 1796 that:

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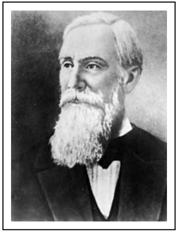

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- This conjectured estimate was proved by Chebyshev in 1848.
- He found explicit constants c, d around 1 such that:

$$\frac{cx}{\ln x} \leq \pi(x) \leq \frac{dx}{\ln x}$$

- Interestingly, using this he was able to show that there is always a prime between n and $2n$, for any $n \geq 2$.

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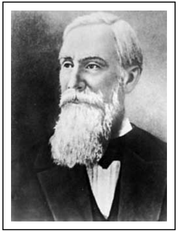

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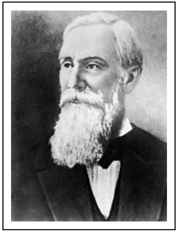

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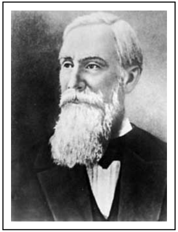

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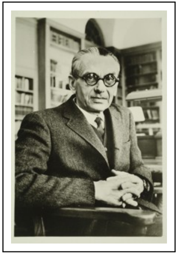

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- Kurt Gödel was probably the first to define the question of *primality testing*, and with it a notion of computational *efficiency* itself.
- In 1956, he asked in a letter to John von Neumann: Can we check whether n is a prime in time polynomial in $\log n$.
- This gave the modern question: Is there a polynomial time algorithm for primality?

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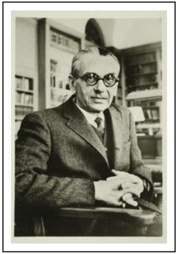

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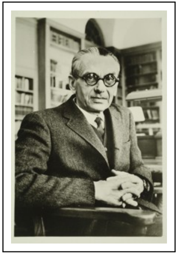

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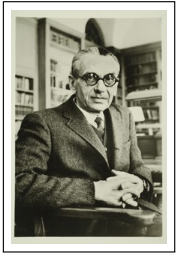

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Can't decide? Toss a coin!

Theorem (Solovay-Strassen, 1977)

An odd number n is prime iff for most a, $a^{\frac{n-1}{2}} = \left(\frac{a}{n}\right) \pmod{n}$.

- Jacobi symbol $\left(\frac{a}{n}\right)$ is computable in time $O^{\sim}(\log^2 n)$.
- We check the above equation for a random a.
- This gives a randomized test that takes time $O^{\sim}(\log^2 n)$.
- It errs with probability at most $\frac{1}{2}$.
- Thus, repeating this process 100 times makes the error probability $\frac{1}{2^{100}}$.

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Theorem (Miller-Rabin, 1980)

An odd number $n = 1 + 2^s \cdot t$ (odd t) is prime iff for most $a \in \mathbb{Z}_n$, the sequence $a^{2^{s-1} \cdot t}, a^{2^{s-2} \cdot t}, \ldots, a^t$ has either a -1 or all 1's.

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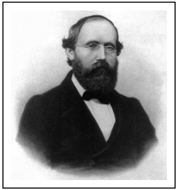

Fig: Riemann

- Can we select the random bits carefully in a randomized algorithm such that there is no error?
- For example, if we assume generalized Riemann Hypothesis (GRH) then the first $(2\log^2 n)$ a's suffice to test primality of n in Solovay-Strassen and Miller-Rabin tests.
- Can we derandomize any randomized polynomial time algorithm?
- Is BPP=P? or

"God does not play dice...."??

Determinism and Randomness: Different?

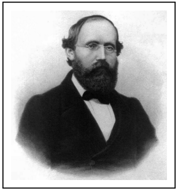

Fig: Riemann

- Can we select the random bits carefully in a randomized algorithm such that there is no error?
- For example, if we assume generalized Riemann Hypothesis (GRH) then the first $(2\log^2 n)$ a's suffice to test primality of n in Solovay-Strassen and Miller-Rabin tests.
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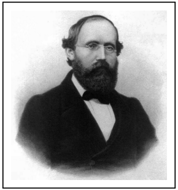

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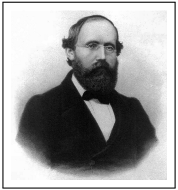

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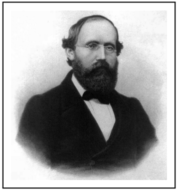

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Determinism and Randomness: Hardness Enters

- In the 1990s it was observed that if there are hard problems then they can be used to derandomize.
- Specifically, Impagliazzo & Wigderson showed in 1997 that BPP=P if E has exponentially hard functions.
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Outline

Primality Testing & Circuits

- Finally, the answer came forth by a rephrasal of primality testing in terms of an *arithmetic circuit*.
- A circuit C over a ring R is a directed acyclic graph with inputs at the leaves, output at the root, $+$ and $*$ as internal nodes, and constants from R at the edges.

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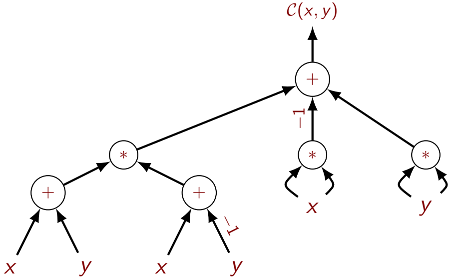

Primality & Zero Circuits

- For any integers $n > 0$ and $1 \leq a \leq n$ define a circuit $C_{n,a}(x) := (x+a)^n - (x^n + a) \pmod{n}$.
- Note that, using repeated squaring, circuit $C_{n,a}$ can be expressed as a directed acyclic graph of size $O(\log n)$.
- It is a simple property of binomial coefficients that:

$$n \text{ is prime iff } C_{n,1}(x) = 0.$$

- It can be viewed as a generalization of Fermat's little theorem.
- It was used by Agrawal & Biswas (1999) to give a new kind of randomized primality test.

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- Although $C_{n,a}(x) := (x+a)^n - (x^n + a) \pmod{n}$ is a $O(\log n)$ sized circuit, checking it for zeroness seems to require computing all the n terms in the expansion of $(x+a)^n$.
- However, if r is "small" we can check $C_{n,a}(x) = 0 \pmod{x^r - 1}$ efficiently.
- Does checking this for few different a & r imply $C_{n,1}(x) = 0$?
- Agrawal, Kayal & Saxena (2002) showed that a, r below $(\log n)^5$ will do!
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Agrawal-Kayal-S Test

1. If n is a^b $(b > 1)$, it is composite.
2. Select an r such that $\mathrm{ord}_r(n) > 4\log^2 n$ and work in the ring $R := \mathbb{Z}_n[x]/(x^r - 1)$.
3. For each a, $1 \leq a \leq \ell := \lceil 2\sqrt{r}\log n \rceil$, check if $(x+a)^n = (x^n + a)$.
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AKS Test: The Proof

- Suppose all the congruences hold and p is a prime factor of n.
- The group $I := \langle n, p \pmod{r} \rangle$. $t := \#I \geq \mathrm{ord}_r(n) \geq 4 \log^2 n$.
- The group $J := \langle (x+1), \ldots, (x+\ell) \pmod{p, h(x)} \rangle$ where $h(x)$ is an irreducible factor of $\frac{x^r-1}{x-1}$ modulo p.

  $\#J \geq 2^{\min\{t,\ell\}} > 2^{2\sqrt{t}\log n} \geq n^{2\sqrt{t}}$.
- *Proof:* Let $f(x), g(x)$ be two different products of $(x+a)$'s, having degree $< t$. Suppose $f(x) = g(x) \pmod{p, h(x)}$.
  - The test tells us that $f(x^{n^i \cdot p^j}) = g(x^{n^i \cdot p^j}) \pmod{p, h(x)}$.
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The Two Groups

Group $I := \langle n, p \pmod{r} \rangle$ is of size $t > 4\log^2 n$.
Group $J := \langle (x+1), \ldots, (x+\ell) \pmod{p, h(x)} \rangle$ is of size $> n^{2\sqrt{t}}$.

- There exist tuples $(i,j) \neq (i',j')$ such that $0 \leq i,j,i',j' \leq \sqrt{t}$ and $n^i \cdot p^j \equiv n^{i'} \cdot p^{j'} \pmod{r}$.
- The test tells us that for all $f(x) \in J$, $f(x)^{n^i \cdot p^j} = f(x^{n^i \cdot p^j})$ and $f(x)^{n^{i'} \cdot p^{j'}} = f(x^{n^{i'} \cdot p^{j'}})$.
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Group $J := \langle (x+1), \ldots, (x+\ell) \pmod{p, h(x)} \rangle$ is of size $> n^{2\sqrt{t}}$.

- There exist tuples $(i,j) \neq (i',j')$ such that $0 \leq i,j,i',j' \leq \sqrt{t}$ and $n^i \cdot p^j \equiv n^{i'} \cdot p^{j'} \pmod{r}$.
- The test tells us that for all $f(x) \in J$, $f(x)^{n^i \cdot p^j} = f(x^{n^i \cdot p^j})$ and $f(x)^{n^{i'} \cdot p^{j'}} = f(x^{n^{i'} \cdot p^{j'}})$.
- Thus, for all $f(x) \in J$, $f(x)^{n^i \cdot p^j} = f(x)^{n^{i'} \cdot p^{j'}}$.
- As J is a cyclic group: $n^i \cdot p^j \equiv n^{i'} \cdot p^{j'} \pmod{\#J}$.
- As $\#J$ is large, $n^i \cdot p^j = n^{i'} \cdot p^{j'}$. Hence, $n = p$ a prime.

AKS Test: The Proof

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AKS Test: Time Complexity

- Recall that r is the least number such that $\text{ord}_r(n) > 4\log^2 n$.
- Prime number theorem gives $r = O(\log^5 n)$ and the algorithm takes time $O^{\sim}(\log^{10.5} n)$.
- Lenstra and Pomerance (2003) further reduced the time complexity to $O^{\sim}(\log^6 n)$.

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Outline

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- However, several modifications have been suggested to AKS test that are faster than the original proposal.
- Can we reduce the number of a for which the test is performed? Here is a conjecture that can bring down the complexity to $O^{\sim}(\log^3 n)$:

Conjecture: (Bhattacharjee-Pandey 2001; AKS 2004)

Let $r > \log n$ be a prime number that does not divide $(n^3 - n)$. Then $(x-1)^n \equiv (x^n - 1) \pmod{n, x^r - 1}$ iff n is prime.

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- Given a circuit $C(x_1, \ldots, x_n)$, determine whether it is the zero circuit in time polynomial in the size of C ??
- Note that AKS primality test solved this question for the special circuit $C(x) = (x+1)^n - (x^n+1) \pmod{n}$.
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