Diagonal Circuit Identity Testing and Lower Bounds

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Outline

The Problem

- Arithmetic circuits, over a field $\mathbb{F}$, compute a polynomial.

- Identity testing is the problem of checking whether a given circuit is zero or not.

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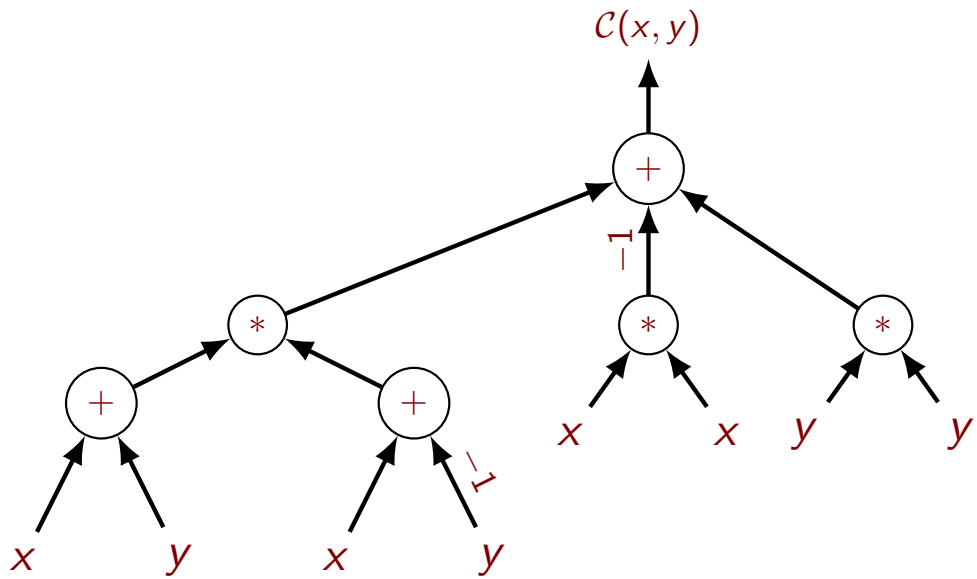

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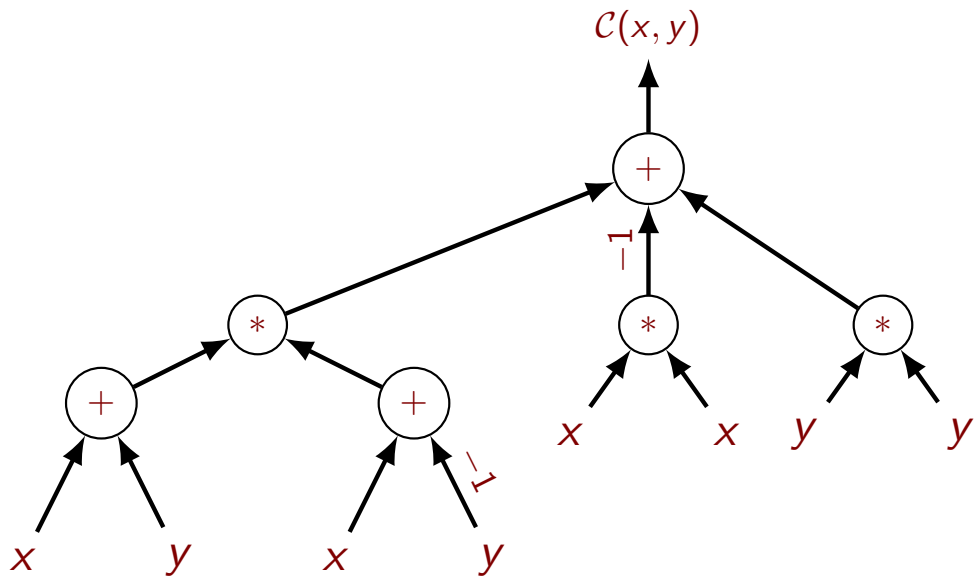

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Motivation

- It is a natural algebraic problem but no efficient algorithm is known.
- Identity testing is instrumental in many results: Parallel algorithms for matching problems (Lovasz '79), PSPACE=IP (Shamir '92), PCP theorem (Arora-Safra '97) and primality testing (AKS '02).
- (Schwartz '80, Zippel '79) gave a randomized algorithm for identity testing.
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- Circuits of depth 3 with bounded top fanin: (Kayal & Saxena '06) gave a deterministic polynomial time identity test.

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The Notation

- For identity testing, wlog we can assume that a depth 4 circuit has a $+$ gate at the top.
- Thus, a depth 4 circuit is a "sum of product of sparse polynomials" ($\Sigma\Pi\Sigma\Pi$ *circuit*).
- Explicitly, a depth-4 circuit C over a field $\mathbb{F}$ will look like:

$$C(x_1, \ldots, x_n) = T_1 + \cdots + T_k$$

where, T_i is a product of polynomials $L_{i,1}, \ldots, L_{i,d}$

where, $L_{i,j}(\bar{x}) = \sum_{\bar{\ell}} a_{i,j,\bar{\ell}} \bar{x}^{\bar{\ell}}$, a's $\in \mathbb{F}$.

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Peculiarity of Depth-4

- Depth-4 is not just another circuit restriction!
- Agrawal & Vinay (FOCS 2008) show that proving exponential lower bounds for depth-4 circuits imply exponential lower bounds for unrestricted depth circuits.
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The Four Ideas

We transform a diagonal circuit $\mathcal{C}$ to a form that is easier to handle. The main ideas are:

(1) We note that the multiplication gate $L_1^{e_1} \dots L_s^{e_s}$ appears in the power series of $exp(zz_1L_1 + \dots + zz_sL_s)$.

(2) We evaluate $exp(zz_1L_1 + \dots + zz_sL_s)$ for various values of z and extract $L_1^{e_1} \dots L_s^{e_s}$ by interpolation.

(3) We transform each multiplication gate to the form above and then stitch them to get a dual form of $\mathcal{C}$: a sum of product of univariates over an algebra.

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- where, $e = (e_1 + \cdots + e_s)$ and E_e is the truncated $\exp(x)$: $1 + x + \cdots + \frac{x^e}{e!}$.

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Idea 2: Interpolation

- We will now extract the degree e part from the above expression.
- If we look at the above sum modulo the ideal $(z_1^{e_1+1}, \ldots, z_s^{e_s+1})$ then the surviving monomial in $\overline{z}$ is exactly $z_1^{e_1} \cdots z_s^{e_s}$.
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- Given a diagonal depth-4 circuit $C(\overline{x}) = T_1 + \cdots + T_k$, where $T_i = \prod_{j=1}^{s} L_{i,j}^{e_{i,j}}$.
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- This sum-of-product-of-univariates circuit is vulnerable!
- We attack it by generalizing the results of Raz & Shpilka (CCC '04) to general commutative algebras.

Thm 1: Let R be an algebra over a field $\mathbb{F}$. Given a sum-of-product-of-univariates circuit $C(x_1, \ldots, x_n)$ over R we can verify deterministically in $poly(size(C), dim(R))$ field operations whether C is zero.

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- The circuit transformation we showed had terms like $\frac{1}{e_1}$, so we need to be careful when the field is of a prime characteristic $p \geq 2$.
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- For a diagonal depth-4 circuit C of the form $\sum_{i=1}^{k} L_{i,1}^{e_{i,1}} \cdots L_{i,s}^{e_{i,s}}$ having *small* s, we gave an identity test and proved lower bounds.
- The main idea was to define a dual form of such circuits.
- Is the dual form useful even when s is variable?

Questions?