Paradigms for *bounded* top-fanin depth-4 circuits

Pranjal Dutta (CMI & IIT Kanpur), **Prateek Dwivedi (IIT K)**, **Nitin Saxena (IIT K)**

CHENNAI MATHEMATICAL INSTITUTE

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

Polynomial Identity Testing

Multivariate Polynomials

- $f(\bar{x}) \in \mathbb{F}[x_1, \dots, x_n]$
- $\deg f = d$. Then, $\sum_j e_j \leq d$.
- $\alpha_{\bar{e}}$ are field elements.

$$f = \sum_{\bar{e}=(e_1,\dots,e_n)} \alpha_{\bar{e}} \cdot \prod_{j\in[n]} x_j^{e_j}$$

$$\begin{aligned} f(x_1, x_2, x_3) &= 1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3 \\ &= (1 + x_1)(1 + x_2)(1 + x_3) \end{aligned}$$

Polynomials: Ubiquitous object in Computer Science

- Graph Algorithm.
- Coding Theory.
- Cryptography.
- Computational Algebra.
- Circuit Complexity.
- Polynomial method in Combinatorics.

Natural Operations

Given a polynomial f,

- Evaluate it at $x_1 = a_1, \ldots, x_n = a_n$.
- For some polynomial g, compute $f + g$ and $f \times g$.
- Find the factors of f.
- *For some polynomial g, test $g = f$.*

Identity Testing

For some polynomial g, test $\mathbf{g} = \mathrm{f}$.

$$f = \sum \alpha_{\bar{e}} \cdot \prod_{j \in [n]} x_j^{e_j}$$

$$g = \sum \beta_{\bar{e}} \cdot \prod_{j \in [n]} x_j^{e_j}$$

- Same coefficients, $\alpha_{\bar{e}} = \beta_{\bar{e}}$?
- Alternatively, check if all coefficients are zero in $f - g$.

That's simple, but not efficient.

Number of coefficients $= \binom{n+d}{d} \approx \mathrm{EXP}(n, d)$.

Representing Multivariate Polynomials

- Sparse Representation: $\overline{\alpha}_{\bar{e}}$.
  - Intuitive.
  - Operations are easy (sum, product, etc)
  - Due to exponential many monomials highly non-succinct.

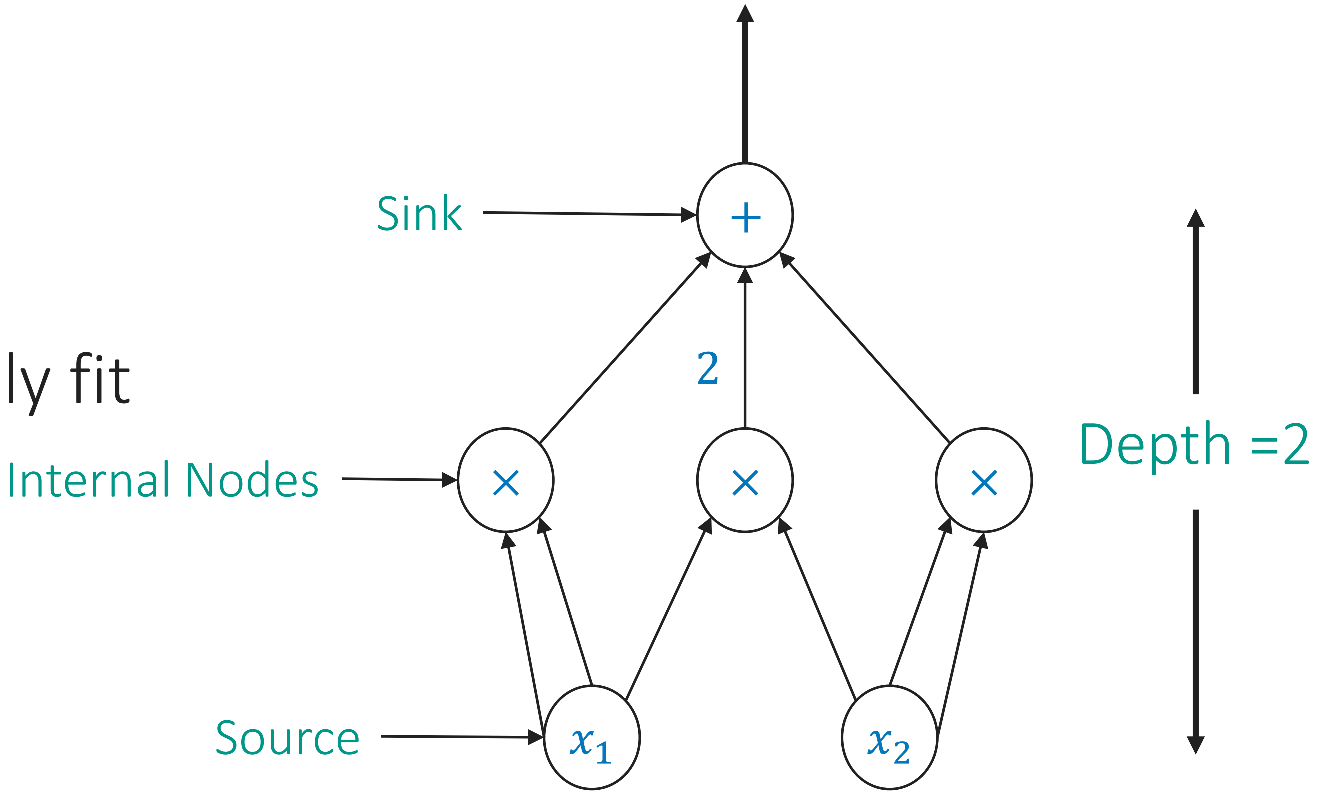
$$f = \sum \alpha_{\bar{e}} \cdot \prod_{j \in [n]} x_j^{e_j}$$

$$f_1 = \sum_{S \subseteq [n]} \prod_{i \in S} x_i \qquad f_2 = \prod_{i = [n]} (x_i + 1)$$

Representing Multivariate Polynomials

- Algebraic Circuits
  - Natural. Succinct.
  - Operations are easy.
  - Algebraic problems naturally fit into the framework.

$$\mathbb{F}[x_1, x_2] \ni \; f_1 = x_1^2 + x_2^2 + 2x_1x_2$$

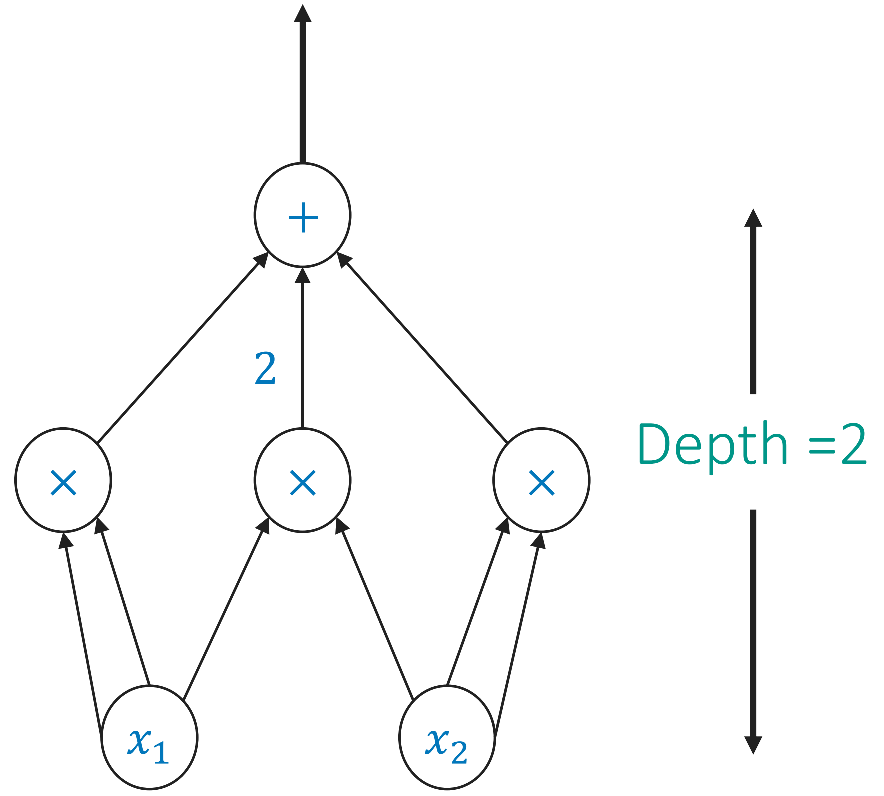


Size = Number of gates = 6

Representing Multivariate Polynomials

- Algebraic Circuits
  - Intuitive. Succinct.
  - Operations are easy.
  - Algebraic problems naturally fit into the framework.
  - PIT is efficient with *randomness*.

$\mathbb{F}[x_1, x_2] \ni \; f_1 = x_1^2 + x_2^2 + 2x_1x_2$



Size = Number of gates = 6

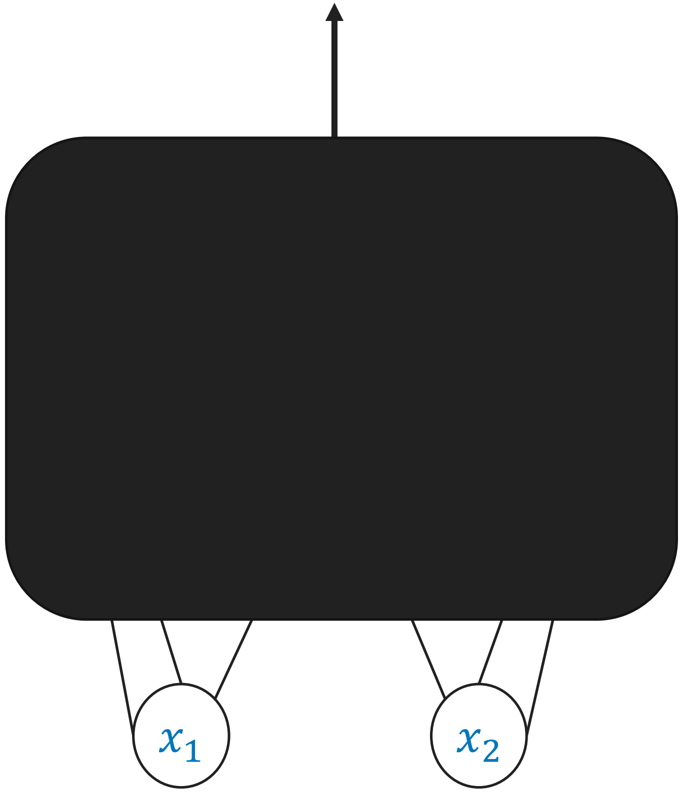
Polynomial Identity Testing

PIT

Given a circuit C over a field $\mathbb{F}$, test if
$C = 0$.

- Whitebox.
- Blackbox.

$$\mathbb{F}[x_1, x_2] \ni \; f_1 = x_1^2 + x_2^2 + 2x_1x_2$$



Efficient Randomized algorithm

PIT Lemma

Let S be a subset of field. For some random $\overline{a} \in S^n$

$$\Pr[\, f(\overline{a}) = 0 \,] \leq \frac{d}{|S|}.$$

- Randomized algorithm: Consider set S of size more than $(d+1)$.
- Also gives a $\mathbf{poly}(d^n)$ time deterministic algorithm.
- Can we do better?

Why do we care?

- Algorithms

PIT ↔ Perfect Matching

For a graph $G(V, E)$ on n verticies,

Tutte Matrix T is a $n \times n$ matrix:

$$T_{ij} = \begin{cases} x_{ij}, \text{if } (i,j) \in E \text{ and } i < j \\ -x_{ij}, \text{if } (i,j) \in E \text{ and } i > j \\ 0, \text{otherwise} \end{cases}$$

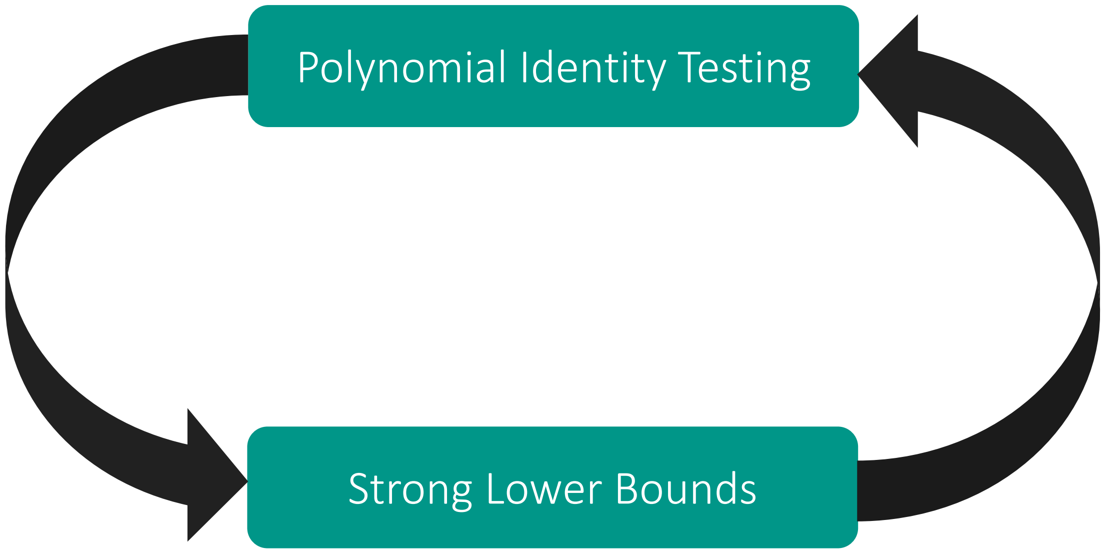
Tutte's Characterisation

G has a perfect matching $\Leftrightarrow \det T \neq 0$

- Determinant is a polynomial. Then testing $det\, T = 0$ is PIT.
- Gives a randomized parallel algorithm using PIT Lemma.

Why do we care?

- Algorithms
- Complexity Theory
- Lower Bounds
  - PIT is intrinsically connected to proving circuit lower bounds.



PIT → Lower Bounds

Kabanets and Impagliazzo (STOC’03)

PIT ∈ P ⟹ Either NEXP is not in P/poly or Permanent is hard.

- “hard” means it requires super polynomial size algebraic circuits.
- Connects derandomizing PIT with Boolean/ Algebraic Lower Bounds.
- Wishful thinking: PIT relates to Permanent hardness?
- Heintz and Schnorr (STOC’80), and later Agrawal (FSTTCS’05), showed PIT ∈ P implies there is a PSPACE computable polynomials which is very hard.

State of Affairs

Status Quo

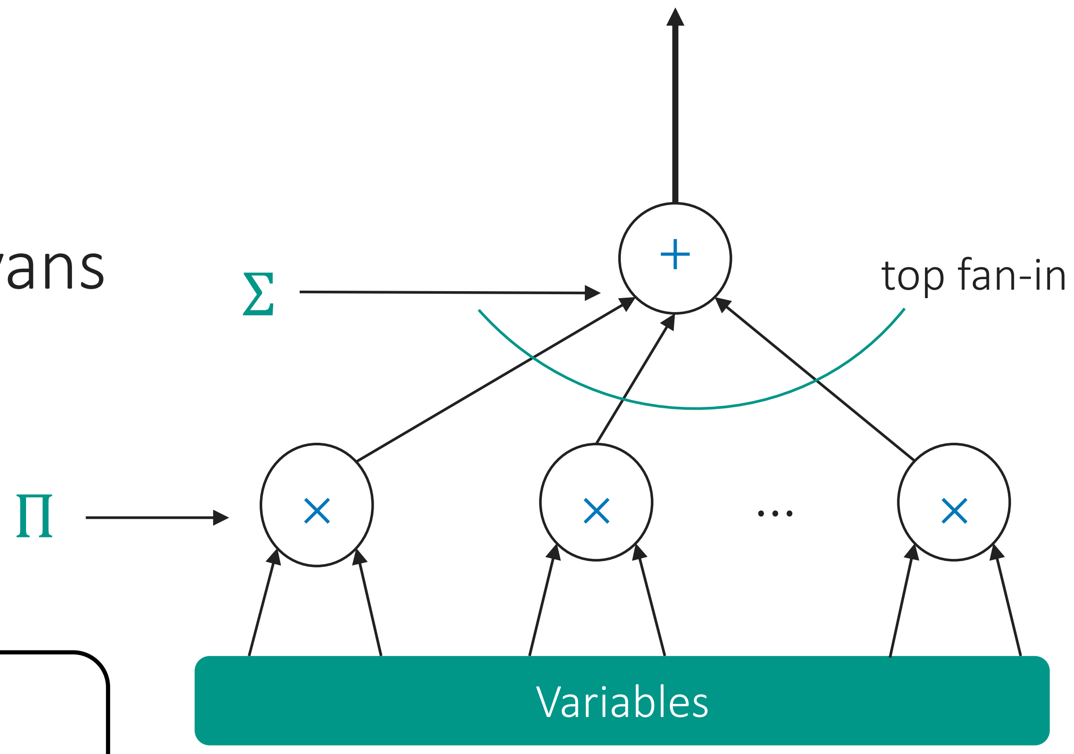
- Nothing better than exponential known for **general** algebraic circuits.
- **Constant depth** circuits in **SUBEXP** algorithm. [Limaye,Srinivasan,Tavenas FOCS’21]
- Efficient algorithm are there for very restricted circuits.

[LST21] Nutan Limaye, Srikanth Srinivasan, Sébastien Tavenas

Depth-2 circuits: Sparse Polynomials

$$\mathbb{F}[x_1, \ldots, x_n] \ni f = \sum_{i}^{poly} (\text{monomial})_i$$

- Monomials only polynomial in n.
- Whitebox is easy.
- Blackbox is easy as well due to Klivans and Spielman (2001).



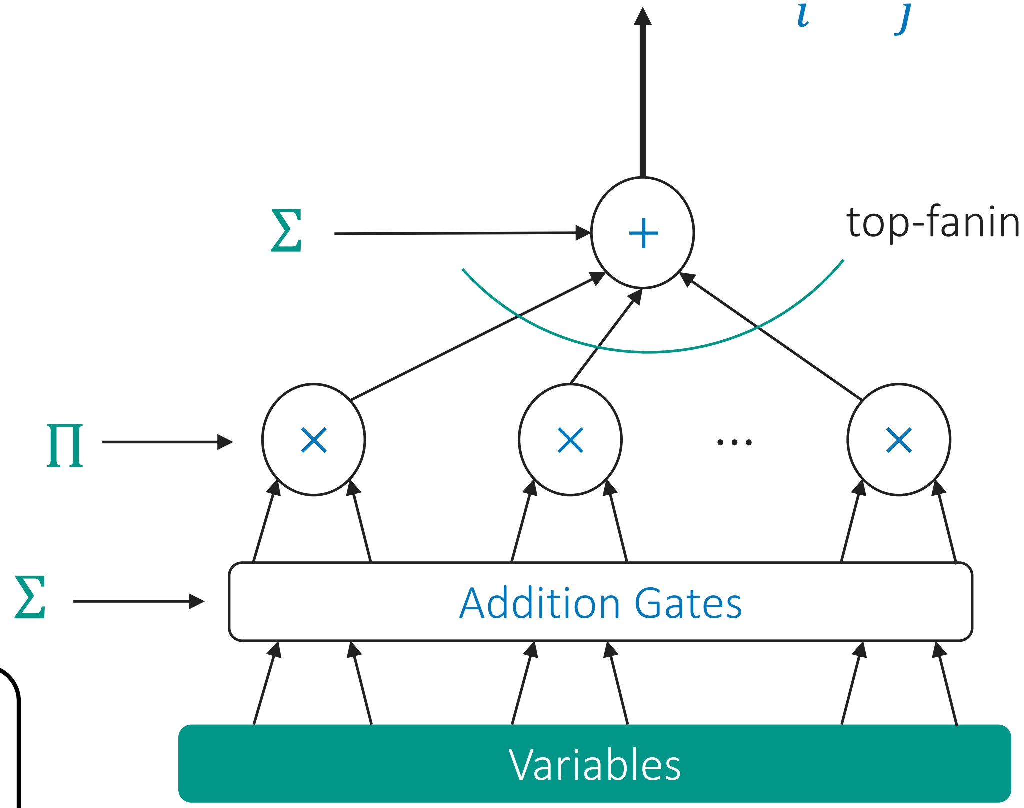
Sparse PIT (KS, STOC'01)

For $\Sigma\Pi$ circuit of size-s and sparsity-m PIT is possible in $\text{poly}(s, m)$.

Depth-3 circuits

- Sum of product of *linear terms*.
- SUBEXP algorithm due to LST21.
- There is poly-time blackbox PIT algorithm when k is constant due to Saxena and Seshadri (2011).

$$\mathbb{F}[x_1, \dots, x_n] \ni f = \sum_i^k \prod_j^d \ell_{ij}$$



$\Sigma^{[k]}\Pi\Sigma$ PIT (SS, STOC'11)

For a size-s circuit PIT algorithm runs in $\mathrm{poly}(s, d^k)$

Depth-4 circuits

$$\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$$

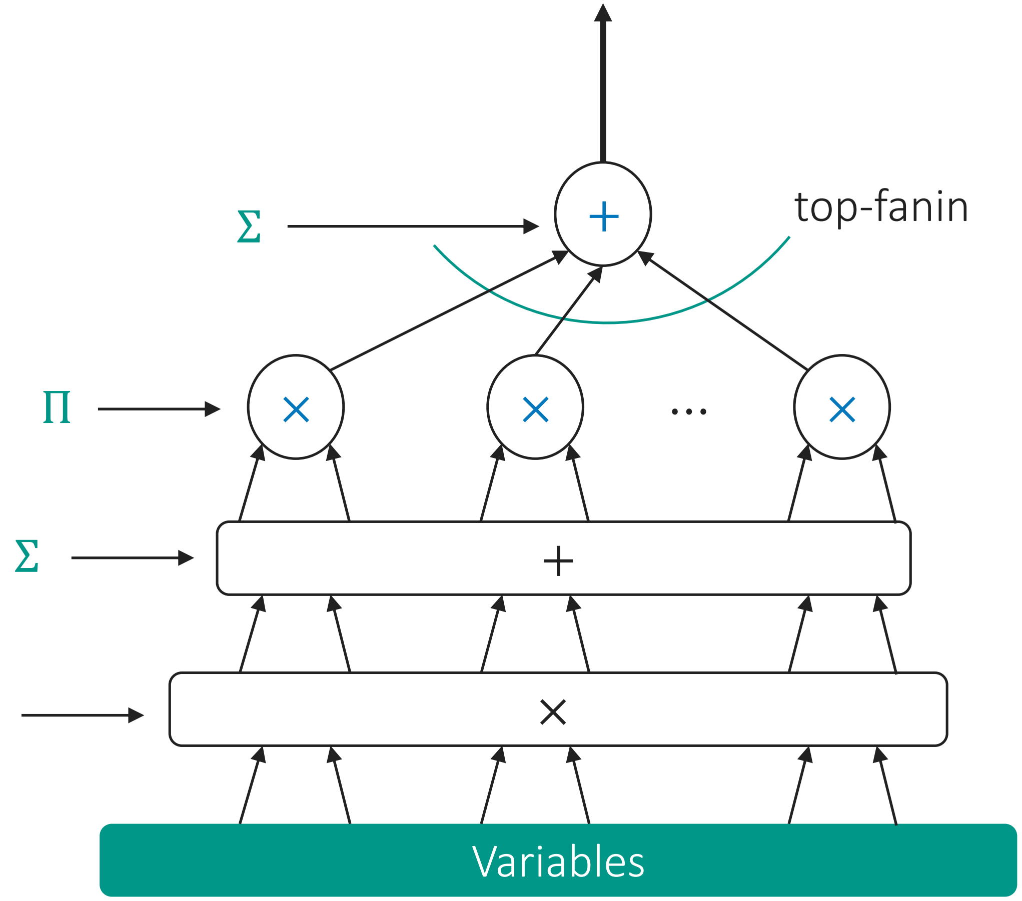
$$\mathbb{F}[x_1, \dots, x_n] \ni f = \sum_{i}^{k} \prod_{j} (\text{sparse polynomial})_{ij}$$

The restriction is special!

Agrawal-Vinay (FOCS'08)

$\Sigma\Pi\Sigma\Pi$ PIT is **almost** as hard as the general case.

- Nothing better than SUBEXP is known.



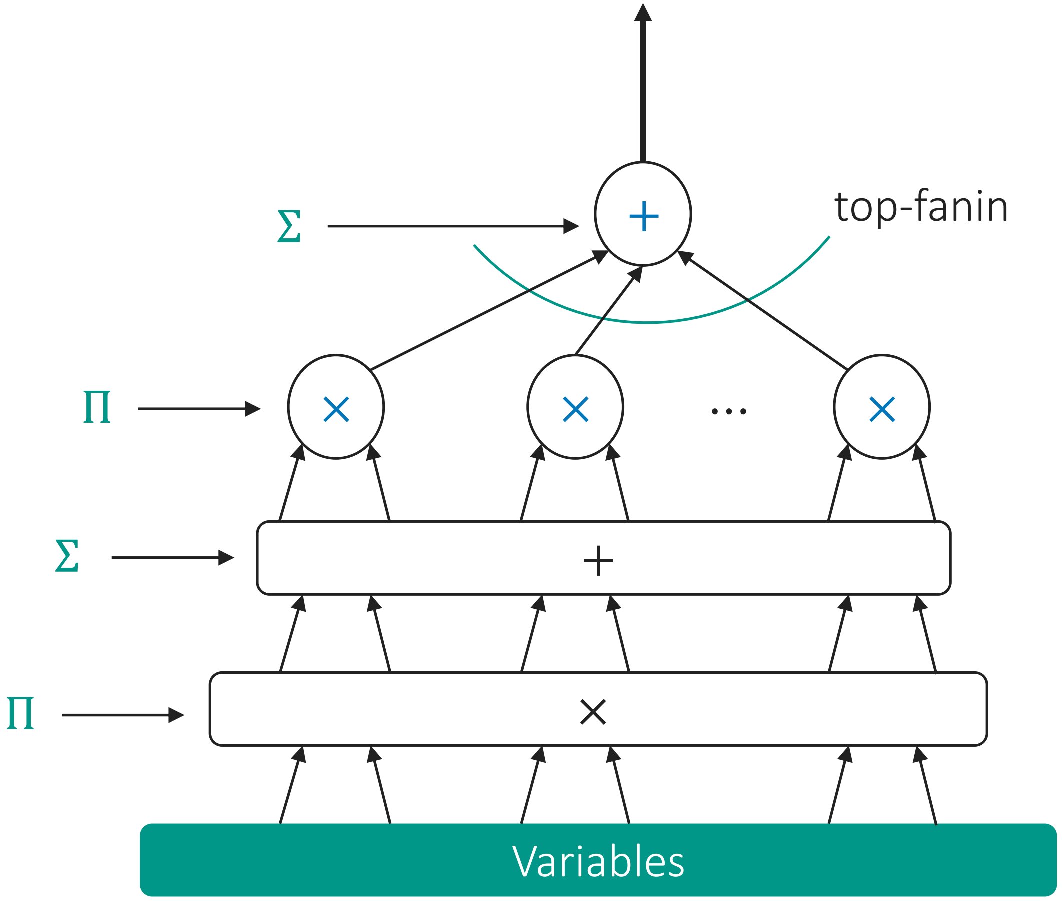
[AV08] Manindra Agrawal, V. Vinay

Depth-4 circuits

$$\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$$

- Promising model.
- Poly (and quasi-poly) time algorithms are found with various *restrictions* on the depth-4 model.

$$\mathbb{F}[x_1, \ldots, x_n] \ni f = \sum_i^k \prod_j (\text{sparse polynomial})_{ij}$$



[AV08] Manindra Agrawal V. Vinay

PIT on Depth Restricted Circuits

$$\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$$

- Promising model.
- Poly (and quasi-poly) time algorithms are found with various *restrictions* on the depth-4 model.
- One such restriction we consider is *bounded top and bottom fanin*.

Paper	Restriction	PIT
Saxena and Seshadhri (STOC'11)	$\delta = 1$	$\text{poly}(n, d^k)$
Beecken, Mittmann and Saxena (ICALP'11)	Bounded trdeg	$\text{poly}(s^k)$ *(k=trdeg bound)*
Agarwal, Saha, Saptharishi and Saxena (STOC'12)	Bounded top-fanin, multilinear	$\text{poly}\left(s^{k^2}\right)$
Kumar and Saraf (CCC'16)	Low individual deg	$\text{QP}(n)$
	Bounded local trdeg and bottom fanin	$\text{QP}(n)$
Peleg and Shpilka (STOC'21)	$k = 3, \delta = 2$	$\text{poly}(n, d)$

New Developments

Blackbox PIT of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits

Theorem 1 [Dutta,Dwivedi,S CCC'21]

For constant k, δ there is a quasi-poly time blackbox PIT algorithm for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits.

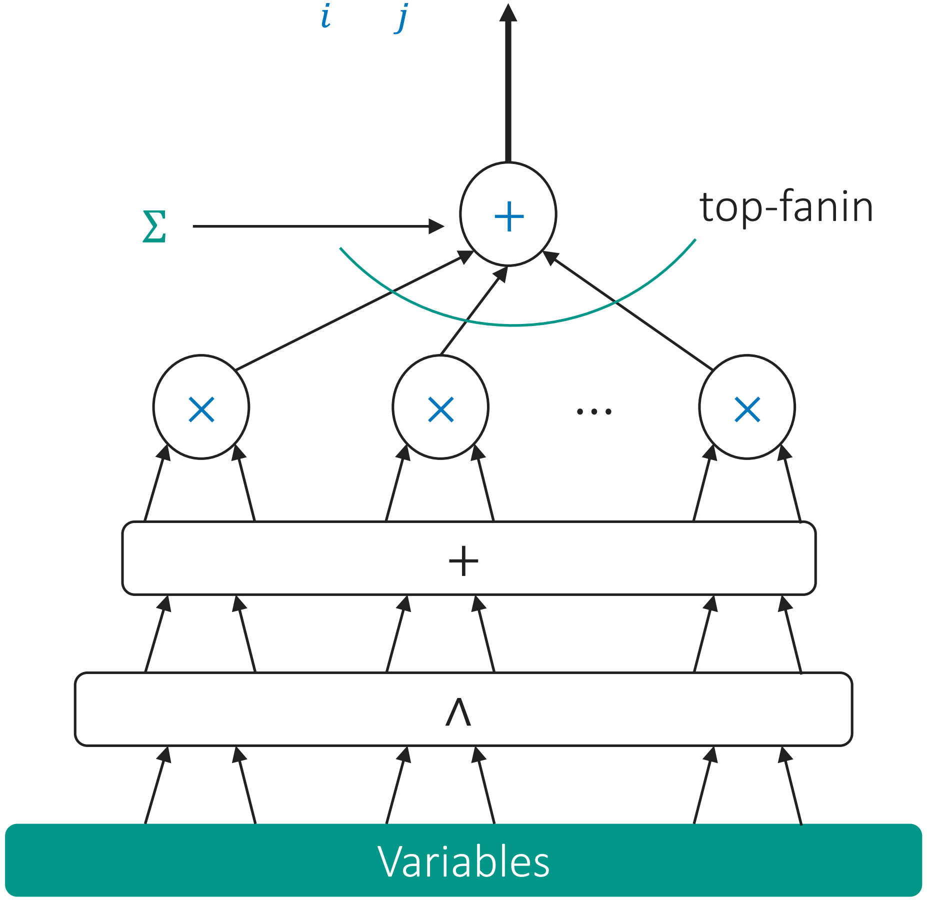
- For size s circuit we give $s^{O(\delta^2 \cdot k \cdot log\ s)}$ time deterministic algorithm.
- The algorithm is **quasi-poly** even up to $k, \delta = \text{poly}(\log s)$.

PIT on $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits

$$\Sigma^{[k]}\Pi\Sigma\wedge$$

- Sum of product of sum of **univariates**.
- Deterministic PIT was open since 2013 [Saha,Saptharishi,S, Comp.Compl.'13].

$$\mathbb{F}[x_1,\ldots,x_n] \ni f = \sum_i^k \prod_j (g_{ij1}(x_1) + \cdots + g_{ijn}(x_n))$$



[SSS13] Chandan Saha, Ramprasad Saptharishi, Nitin Saxena

Blackbox PIT of $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits

Theorem 2 [Dutta,Dwivedi,S CCC'21]

For constant k there is a quasi-poly time blackbox PIT algorithm for $\Sigma^{[\mathrm{k}]}\Pi\Sigma\wedge$ circuits.

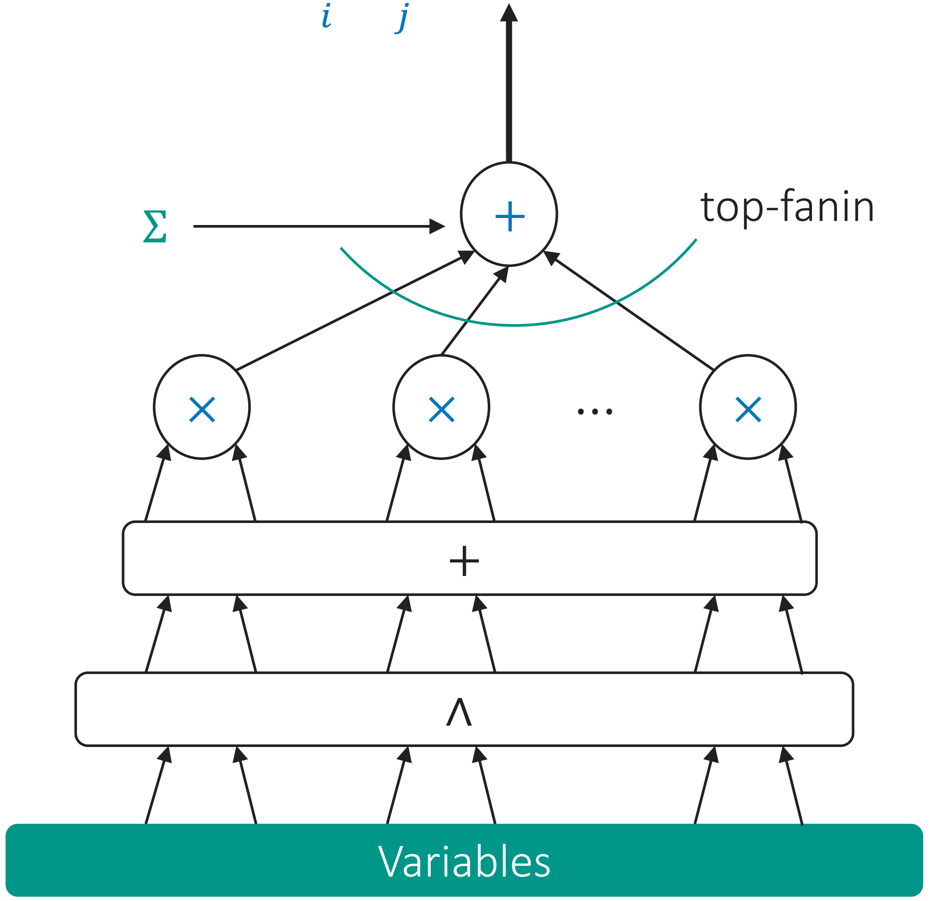
- For size s circuit we take $s^{O(k\cdot log\ log\ s)}$ time.
- Similar proof, but faster than our $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ PIT algo.

Whitebox PIT on $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits

$$\Sigma^{[k]}\Pi\Sigma\wedge$$

- Sum of product of sum of **univariates**.
- $k \leq 2$ was already solved in [SSS13].
- But $k > 2$ was open!

$$\mathbb{F}[x_1, \dots, x_n] \ni f = \sum_{i}^{k} \prod_{j} \left(g_{ij1}(x_1) + \cdots + g_{ijn}(x_n)\right)$$



[SSS13] Chandan Saha, Ramprasad Saptharishi, Nitin Saxena

Whitebox PIT of $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits

Theorem 3 [Dutta,Dwivedi,S CCC'21]

For constant k there is a poly time whitebox PIT algorithm for $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits.

- For size s circuit we take $s^{O(k\cdot 7^k)}$ time.
- Introducing DiDI-technique (Divide Derive Induct).
  - Inductive. Top $\Pi \to \wedge$.
  - Robust enough to give blackbox algorithm. But worse time complexity.

Conclusion

Conclusion

- Introduced PIT and Algebraic Circuits.
- Discussed connection of PIT with algorithms and lower bounds.
- Three new PIT algorithms: Blackbox PIT of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ and $\Sigma^{[k]}\Pi\Sigma\wedge$. And Whitebox PIT of $\Sigma^{[k]}\Pi\Sigma\wedge$.

Open Problems

- Design a poly-time algorithm for $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuits.
  - It will place PIT of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ in P.
- Solve PIT for $\Sigma^{[k]}\Pi\Sigma\wedge^{[2]}$ - sum of product of sum of bivariate fed into top product gate.
- Improve the dependence on k for $\Sigma^{[k]}\Pi\Sigma\wedge$ whitebox PIT.
  - Currently it is exponential in k.

Proof Overview

- **Now**: Jacobian for blackbox PIT

- *Monday: Alternate approach (DiDI)*

Recapitulation of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ blackbox PIT

Problem

Test

$$f = T_1 + T_2 + \cdots T_k \stackrel{?}{=} 0$$

Where $T_i = \prod_j g_{ij} \in \Pi\Sigma\Pi^{[\delta]}$ of degree at most d and size s.

Design a homomorphism
$\Psi: \mathbb{F}[\bar{x}] \to \mathbb{F}[\bar{x}, z]$
manifesting nice property.

↓

Fixing $\bar{x}$ suitably using 'nice' property
$\Psi': \mathbb{F}[\bar{x}] \to \mathbb{F}[z]$
such that it preserves rank of Jacobian.

↓

Extend Ψ' to a faithful map
$\Phi: \mathbb{F}[\bar{x}] \to \mathbb{F}[z, \bar{y}, t]$

↓

Use PIT Lemma for final Hitting Set of
$\Phi(f)$

Hitting Set

Definition [Hitting Set]

A set $\mathcal{H}$ which certifies the non-zeroness of class $\mathcal{C}$ of polynomials.

$$\forall\, f \neq 0 \in \mathcal{C}, \qquad \exists \bar{a} \in \mathcal{H} : f(\bar{a}) \neq 0$$

- Blackbox PIT $\leftrightarrow$ Hitting Set.

Trivial Hitting Set

Lemma [Trivial Hitting Set]

For a class of n-variate, deg d polynomials, there exists an explicit hitting set of size $\mathbf{poly}(d^n)$

- Suffices when $n = O(1)$.
- Offers a general framework for PIT algorithms.
  - Design a variable reducing non-zeroness preserving map.

Recapitulation of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ blackbox PIT

Problem

Test

$$f = T_1 + T_2 + \cdots T_k \stackrel{?}{=} 0$$

Where $T_i = \prod_j g_{ij} \in \Pi\Sigma\Pi^{[\delta]}$ of degree at most d and size s.

Design a homomorphism
$\Psi: \mathbb{F}[\bar{x}] \to \mathbb{F}[\bar{x}, z]$
manifesting nice property.

Fixing $\bar{x}$ suitably using 'nice' property
$\Psi': \mathbb{F}[\bar{x}] \to \mathbb{F}[z]$
such that it preserves rank of Jacobian.

Extend Ψ' to a faithful map
$\Phi: \mathbb{F}[\bar{x}] \to \mathbb{F}[z, \bar{y}, t]$

Use PIT Lemma for final Hitting Set of
$\Phi(f)$

Faithful homomorphism

- Set of polynomials $\bar{T} = \{T_1, \ldots, T_m\}$ in $\mathbb{F}[\bar{x}]$ are *algebraically dependent* if there is an non-zero *annihilator* A such that $A(\bar{T}\ ) = 0$.
- Transcendence Degree (trdeg): Size of the largest subset of $S \subseteq \bar{T}$ which is alg. independent.
  - S is called the *Transcendence Basis*.

Faithful homomorphism

Definition [Faithful hom.]

$\Phi: \mathbb{F}[\bar{x}] \to \mathbb{F}[\bar{y}]$ such that

$$\mathrm{trdeg}_{\mathbb{F}}\,(\bar{T}) = \mathrm{trdeg}_{\mathbb{F}}\big(\Phi(\bar{T})\big).$$

Theorem [Faithful is useful]

For any $C \in \mathbb{F}[y_1, \ldots, y_m]$,

$$C(\bar{T}) = 0 \iff C\big(\Phi(\bar{T})\big) = 0.$$

Recapitulation of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ blackbox PIT

Problem

Test

$$f = T_1 + T_2 + \cdots T_k \stackrel{?}{=} 0$$

Where $T_i = \prod_j g_{ij} \in \Pi\Sigma\Pi^{[\delta]}$ of degree at most d and size s.

Design a homomorphism
$\Psi: \mathbb{F}[\bar{x}] \to \mathbb{F}[\bar{x}, z]$
manifesting '*nice*' property.

↓

Fixing $\bar{x}$ suitably using 'nice' property
$\Psi': \mathbb{F}[\bar{x}] \to \mathbb{F}[z]$
such that it preserves rank of *Jacobian*.

↓

Extend Ψ' to a *faithful* map
$\Phi: \mathbb{F}[\bar{x}] \to \mathbb{F}[z, \bar{y}, t]$

↓

Use PIT Lemma for final Hitting Set of
$\Phi(f)$

Jacobian Hits (Again)

- Jacobian $\mathcal{J}_{\bar{x}}(\bar{T})$ is a $m \times n$ matrix.

$$\mathcal{J}_{\bar{x}}(\bar{T}) = \left(\partial_{x_j}(T_i)\right)_{m\times n} = \begin{bmatrix} \partial_{x_1}(T_1) & \cdots & \partial_{x_n}(T_1) \\ \vdots & \ddots & \vdots \\ \partial_{x_1}(T_m) & \cdots & \partial_{x\_n}(T_m) \end{bmatrix}$$

- Linear rank captures the alg. rank.

Theorem [Beecken,Mittmann,Saxena, ICALP'11]

Jacobian Criterion: For large $\mathrm{char}\ \mathbb{F}$,

$$\mathrm{trdeg}_{\mathbb{F}}\ (\bar{T}) = \mathrm{rank}_{\mathbb{F}(\bar{x})}\mathcal{J}_{\bar{x}}(\bar{T})$$

Jacobian Hits (Again)

- Jacobian offers the recipe of *faithful* map.
- Let $\Psi' : \mathbb{F}[\bar{x}] \to \mathbb{F}[\bar{z}]$ such that

$$\mathrm{rank}_{\mathbb{F}(\bar{x})} \mathcal{J}_{\bar{x}}(\bar{T}) = \mathrm{rank}_{\mathbb{F}(\bar{z})} \Psi'\left(\mathcal{J}_{\bar{x}}(\bar{T})\right).$$

Theorem [ASSS, STOC'12*]

For large $\mathbf{char}\,\mathbb{F}$, the map $\Phi: \mathbb{F}[\bar{x}] \to \mathbb{F}[z, \bar{y}, t]$ *defined as*

$$x_i \to \left(\sum_{j \le k} y_j\, t^{ij}\right) + \Psi'(x_i)$$

is *faithful* for $T_1, \ldots T_k$.

**Agarwal, Saha, Saptharishi and Saxena*

Recapitulation of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ blackbox PIT

Problem

Test

$$f = T_1 + T_2 + \cdots T_k \stackrel{?}{=} 0$$

Where $T_i = \prod_j g_{ij} \in \Pi\Sigma\Pi^{[\delta]}$ of degree at most d and size s.

Design a homomorphism
$\Psi: \mathbb{F}[\bar{x}] \to \mathbb{F}[\bar{x}, z]$
manifesting '*nice*' property.

↓

Fixing $\bar{x}$ suitably using 'nice' property
$\Psi': \mathbb{F}[\bar{x}] \to \mathbb{F}[z]$
such that it preserves rank of *Jacobian*.

↓

Extend Ψ' to a *faithful* map
$\Phi: \mathbb{F}[\bar{x}] \to \mathbb{F}[z, \bar{y}, t]$

↓

Use PIT Lemma for final Hitting Set of
$\Phi(f)$

Homomorphism Ψ

- Suppose $T_1, \ldots, T_k$ is the tr-basis.

$$\mathcal{J}_{\bar{x}}(\bar{T}) := \left(\partial_{x_j}(T_i)\right)_{k\times k}$$

- Let $J_{\bar{x}}(\bar{T}) = \operatorname{Det} \mathcal{J}_{\bar{x}}(\bar{T})$,
  - To preserve rank, ensure determinant is non-zero.
  - $L(T_i) := \{g_{ij} \mid j\}$.

$$J_{\bar{x}}(\bar{T}) = T_1 \ldots T_k \sum_{g_1\in L(T_1),\ldots g_k\in L(T_k)} \frac{J_{\bar{x}}(g_1, \ldots, g_k)}{g_1 \cdots g_k}$$

Homomorphism Ψ

- Consider an $\bar{\alpha} = (a_1, \dots, a_n) \subseteq \mathbb{F}^n$ such that $g(\bar{\alpha}) \neq 0$ for all $g \in \bigcup_i L(T_i)$. Find it using PIT for sparse polynomials.
- Define $\Psi: \mathbb{F}[\bar{x}] \to \mathbb{F}[\bar{x}, z]$ such that

$$x_i \mapsto z \cdot x_i + a_i.$$

$$\Psi(J_{\bar{x}}(\bar{T})) = \Psi(T_1 \dots T_k) \underbrace{\boxed{\sum_{(\cdot)} \frac{\Psi(J_{\bar{x}}(g_1, \dots, g_k))}{\Psi(g_1 \cdots g_k)}}}_{\mathrm{F}}$$

Homomorphism Ψ

For *inverses*--- Define $\mathcal{R} := \mathbb{F}[z]/\langle z^D \rangle$, where $D := k(d-1)+1$.

Claim

Over $\mathcal{R}[\bar{x}]$,

- $J_{\bar{x}}(\bar{T}) = 0 \iff \Psi(J_{\bar{x}}(\bar{T})) = 0.$
- $\Psi\big(J_{\bar{x}}(\bar{T})\big) = 0 \iff F = 0.$

- Wlog assume $J_{\bar{x}}(\bar{T}) \neq 0$, then $F \neq 0$ over $\mathcal{R}[\bar{x}]$.
- Construct a set $H' \subseteq \mathbb{F}^n$: $\Psi\big(J_{\bar{x}}(\bar{T})\big)\big|_{\bar{x}=\bar{a}} \neq 0$, for some $\bar{a} \in H'$.
- For this we construct a hitting-set for F.

Recapitulation of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ blackbox PIT

Problem

Test

$$f = T_1 + T_2 + \cdots T_k \stackrel{?}{=} 0$$

Where $T_i = \prod_j g_{ij} \in \Pi\Sigma\Pi^{[\delta]}$ of degree at most d and size s.

Design a homomorphism
$\Psi: \mathbb{F}[\bar{x}] \to \mathbb{F}[\bar{x}, z]$
manifesting '*nice*' property.

↓

Fixing $\bar{x}$ suitably using 'nice' property
$\Psi': \mathbb{F}[\bar{x}] \to \mathbb{F}[z]$
such that it preserves rank of *Jacobian*.

↓

Extend Ψ' to a *faithful* map
$\Phi: \mathbb{F}[\bar{x}] \to \mathbb{F}[z, \bar{y}, t]$

↓

Use PIT Lemma for final Hitting Set of $\Phi(f)$

Towards extending Ψ to Ψ'

Claim [Nice Property]

Over $\mathcal{R}[\bar{x}]$, F can be computed by $\Sigma \wedge \Sigma\Pi^{[\delta]}$-circuit of size $\left(s \cdot 3^{\delta}\right)^{O(k)}$.

- $F := P(\bar{x}, z)/Q$, where $Q \in \mathbb{F}$.
- Degree of P wrt z remains polynomially bounded.

$\Sigma \wedge \Sigma\Pi^{[\delta]}$ - sum of powers of (degree δ) sparse polynomials.

Towards extending Ψ to Ψ'

- *Essentially,* H' will be the hitting-set for ‘small’ size $\Sigma \wedge \Sigma\Pi^{[\delta]}$.
- [Forbes, FOCS’15] gave the hitting set for the class.
- Use that to conclude that $H' \subseteq \mathbb{F}^n$ such that $P(H', z) \neq 0$ is of size $s^{O(\delta^2 \cdot k \cdot \log s)}$.
- H' fixes $\bar{x}$ in Ψ and gives Ψ'.

Recapitulation of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ blackbox PIT

Problem

Test

$$f = T_1 + T_2 + \cdots T_k \stackrel{?}{=} 0$$

Where $T_i = \prod_j g_{ij} \in \Pi\Sigma\Pi^{[\delta]}$ of degree at most d and size s.

- Construction of faithful map Φ follows from Hitting set of $\Sigma \wedge \Sigma\Pi^{[\delta]}$-circuit.
- Therefore, $\Phi(f)$ is essentially $k+3$ variate polynomial.

Design a homomorphism $\Psi: \mathbb{F}[\bar{x}] \to \mathbb{F}[\bar{x}, z]$ manifesting '*nice*' property.

↓

Fixing $\bar{x}$ suitably using 'nice' property $\Psi': \mathbb{F}[\bar{x}] \to \mathbb{F}[z]$ such that it preserves rank of *Jacobian*.

↓

Extend Ψ' to a *faithful* map $\Phi: \mathbb{F}[\bar{x}] \to \mathbb{F}[z, y_1, \ldots y_k, t]$

↓

Use PIT Lemma for final Hitting Set of $\Phi(f)$

Open Problems

- Design a *poly-time* algorithm for $\Sigma \wedge \Sigma\Pi^{[\delta]}$-circuits?
  - It will place PIT of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ in P.
- Solve PIT for $\Sigma^{[k]}\Pi\Sigma \wedge^{[2]}$ - sum of product of sum of bivariate fed into top product gate?
- Improve the dependence on k for $\Sigma^{[k]}\Pi\Sigma \wedge$ whitebox PIT?
  - Currently it is exponential in k.