Prime Numbers and Circuits

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1. Brief History of Primes
2. Primality testing
3. Derandomization?
4. Circuits
5. Primality Derandomized
6. Questions

Outline

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Prime Numbers

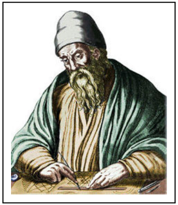

Fig: Euclid

- An integer $n > 1$ is *prime* if its divisors are only 1 and n.
- They are the building blocks of numbers and this means, as Euclid demonstrated in 300 B.C., primes are infinitely many.
- Not only are they pervasive in Mathematics but also appear in practice eg. Cryptography, Communication, ...
- So how do we check and find primes?

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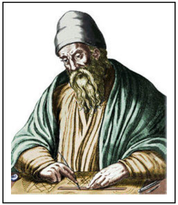

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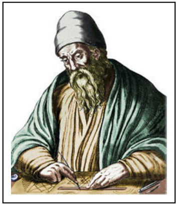

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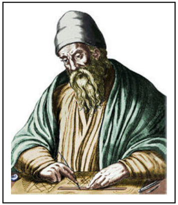

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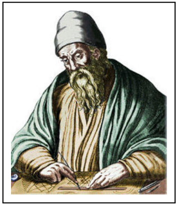

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Eratosthenes & his sieve

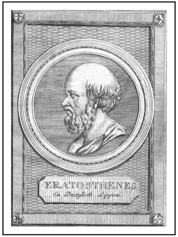

Fig: Eratosthenes

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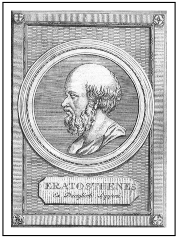

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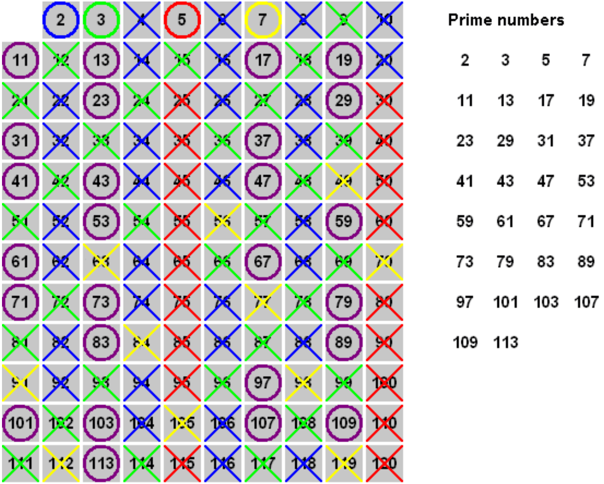

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Eratosthenes Sieve

"Sift the Twos and sift the Threes, The Sieve of Eratosthenes. When the multiples sublime, The numbers that remain are Prime."

- This is the high school method to test primes, attributed to Eratosthenes 200 B.C.
- For a number n, it is sufficient to divide by numbers upto $\sqrt{n}$.
- Thus, it takes around $O(\sqrt{n})$ steps. For a 100-bit number this means 2^{50} steps!

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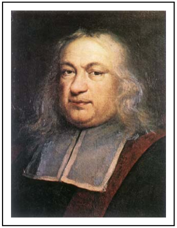
Fermat & his Little Theorem

Theorem (Fermat, 1660s)

If n is prime then for every a, $a^n = a \pmod n$.



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- It is easy to compute $a^n \pmod n$ using repeated squaring (i.e. compute sequentially $a \pmod n$, $a^2 \pmod n$, $a^4 \pmod n$,...) this takes time $\log^2 n$, which for a 100-bit number is only 100^2 steps.
- Can we ascertain the primality of n by checking $a^n = a \pmod n$ for few *magical* a?
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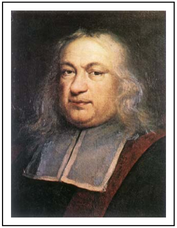

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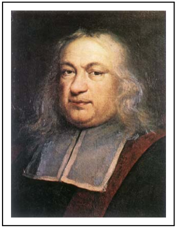

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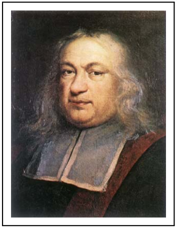

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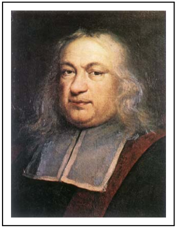

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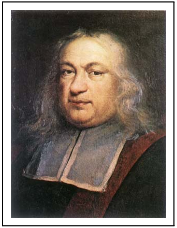

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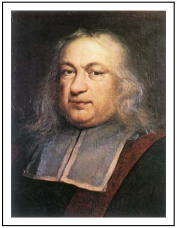

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Prime Number Estimates

Fig: Gauss

- For any real $x > 1$, let $\pi(x)$ be the number of primes $p \leq x$.
- By looking at the tables of primes Legendre and Gauss (independently) conjectured in 1796 that:

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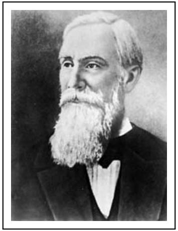

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- This conjectured estimate was proved by Chebyshev in 1848.
- He found explicit constants c, d around 1 such that:

$$\frac{cx}{\ln x} \leq \pi(x) \leq \frac{dx}{\ln x}$$

- Interestingly, using this he was able to show that there is always a prime between n and $2n$, for any $n \geq 2$.

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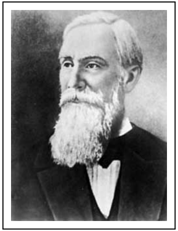

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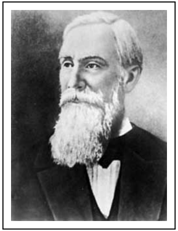

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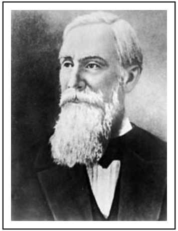

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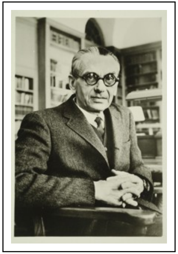

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- Kurt Gödel was probably the first to define the question of *primality testing*, and with it a notion of computational *efficiency* itself.
- In 1956, he asked in a letter to John von Neumann: Can we check whether n is a prime in time polynomial in $\log n$.
- This gave the modern question: Is there a polynomial time algorithm for primality?

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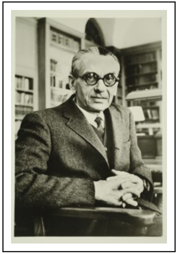

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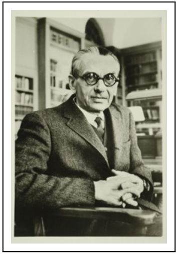

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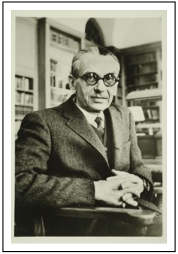

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Can't decide? Toss a coin!

Theorem (Solovay-Strassen, 1977)

An odd number n is prime iff for most a, $a^{\frac{n-1}{2}} = \left(\frac{a}{n}\right) \pmod{n}$.

- Jacobi symbol $\left(\frac{a}{n}\right)$ is computable in time $O^{\sim}(\log^2 n)$.
- We check the above equation for a random a.
- This gives a randomized test that takes time $O^{\sim}(\log^2 n)$.
- It errs with probability at most $\frac{1}{2}$.
- Thus, repeating this process 100 times makes the error probability $\frac{1}{2^{100}}$.

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Primality: A Practical Solution

Theorem (Miller-Rabin, 1980)

An odd number $n = 1 + 2^s \cdot t$ (odd t) is prime iff for most $a \in \mathbb{Z}_n$, the sequence $a^{2^{s-1}\cdot t}, a^{2^{s-2}\cdot t}, \ldots, a^t$ has either a -1 or all 1's.

- We check the above condition for a random a.
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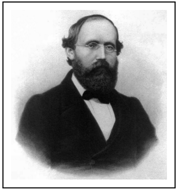

Fig: Riemann

- Can we select the random bits carefully in a randomized algorithm such that there is no error?
- For example, if we assume generalized Riemann Hypothesis (GRH) then the first $(2\log^2 n)$ a's suffice to test primality of n in Solovay-Strassen and Miller-Rabin tests.
- Can we derandomize any randomized polynomial time algorithm?
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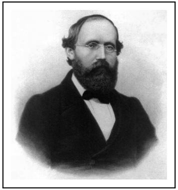

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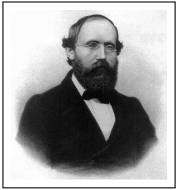

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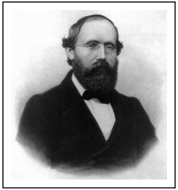

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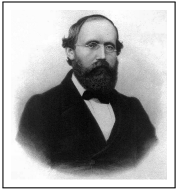

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Determinism and Randomness: Hardness Enters

- In the 1990s it was observed that if there are hard problems then they can be used to derandomize.
- Specifically, Impagliazzo & Wigderson showed in 1997 that BPP=P if E has exponentially hard functions.
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2. Primality testing
3. Derandomization?
4. Circuits
5. Primality Derandomized
6. Questions

PRIMALITY TESTING & CIRCUITS

- Finally, the answer came forth by a rephrasal of primality testing in terms of an *arithmetic circuit*.
- A circuit $\mathcal{C}$ over a ring R is a directed acyclic graph with inputs at the leaves, output at the root, $+$ and $*$ as internal nodes, and constants from R at the edges.

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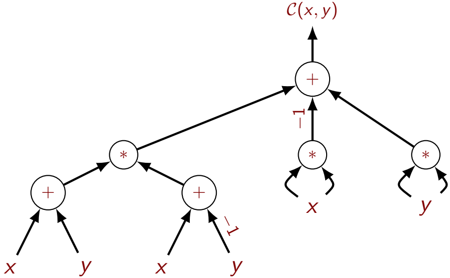

Primality & Zero Circuits

- For any integers $n > 0$ and $1 \leq a \leq n$ define a circuit $C_{n,a}(x) := (x+a)^n - (x^n + a) \pmod{n}$.
- Note that, using repeated squaring, circuit $C_{n,a}$ can be expressed as a directed acyclic graph of size $O(\log n)$.
- It is a simple property of binomial coefficients that:

$$n \text{ is prime iff } C_{n,1}(x) = 0.$$

- It can be viewed as a generalization of Fermat's little theorem.
- It was used by Agrawal & Biswas (1999) to give a new kind of randomized primality test.

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The Idea

- Although $C_{n,a}(x) := (x+a)^n - (x^n+a) \pmod n$ is a $O(\log n)$ sized circuit, checking it for zeroness seems to require computing all the n terms in the expansion of $(x+a)^n$.
- However, if r is "small" we can check $C_{n,a}(x) = 0 \pmod{x^r - 1}$ efficiently.
- Does checking this for few different a & r imply $C_{n,1}(x) = 0$?
- Agrawal, Kayal & Saxena (2002) showed that a, r below $(\log n)^5$ will do!
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1. If n is a^b ($b > 1$), it is composite.
2. Select an r such that $\text{ord}_r(n) > 4\log^2 n$ and work in the ring $R := \mathbb{Z}_n[x]/(x^r - 1)$.
3. For each a, $1 \leq a \leq \ell := \lceil 2\sqrt{r}\log n \rceil$, check if $(x+a)^n = (x^n + a)$.
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- Suppose all the congruences hold and p is a prime factor of n.
- The group $I := \langle n, p \pmod{r} \rangle$. $t := \#I \geq \mathrm{ord}_r(n) \geq 4\log^2 n$.
- The group $J := \langle (x+1), \ldots, (x+\ell) \pmod{p, h(x)} \rangle$ where $h(x)$ is an irreducible factor of $\frac{x^r-1}{x-1}$ modulo p.
  $\#J \geq 2^{\min\{t,\ell\}} > 2^{2\sqrt{t}\log n} \geq n^{2\sqrt{t}}$.
- *Proof:* Let $f(x), g(x)$ be two different products of $(x+a)$'s, having degree $< t$. Suppose $f(x) = g(x) \pmod{p, h(x)}$.
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The Two Groups

Group $I := \langle n, p \pmod r \rangle$ is of size $t > 4\log^2 n$.
Group $J := \langle (x+1), \ldots, (x+\ell) \pmod{p, h(x)} \rangle$ is of size $> n^{2\sqrt{t}}$.

- There exist tuples $(i,j) \neq (i',j')$ such that $0 \leq i,j,i',j' \leq \sqrt{t}$ and $n^i \cdot p^j \equiv n^{i'} \cdot p^{j'} \pmod r$.
- The test tells us that for all $f(x) \in J$, $f(x)^{n^i \cdot p^j} = f(x^{n^i \cdot p^j})$ and $f(x)^{n^{i'} \cdot p^{j'}} = f(x^{n^{i'} \cdot p^{j'}})$.
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- As $\#J$ is large, $n^i \cdot p^j = n^{i'} \cdot p^{j'}$. Hence, $n = p$ a prime.

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AKS Test: The Proof

The Two Groups

Group $I := \langle n, p \pmod{r} \rangle$ is of size $t > 4\log^2 n$.
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AKS Test: Time Complexity

- Recall that r is the least number such that $\text{ord}_r(n) > 4\log^2 n$.
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Outline

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2. Primality testing
3. Derandomization?
4. Circuits
5. Primality Derandomized
6. Questions

Questions

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Let $r > \log n$ be a prime number that does not divide $(n^3 - n)$. Then $(x-1)^n \equiv (x^n - 1) \pmod{n, x^r - 1}$ iff n is prime.

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- Note that AKS primality test solved this question for the special circuit $C(x) = (x+1)^n - (x^n + 1) \pmod{n}$.
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