CS748 - ARITHMETIC CIRCUIT COMPLEXITY
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ASSIGNMENT 3

POINTS: 50

DATE GIVEN: 27-SEP-2024 DUE: 20-OCT-2024

Rules:
- You are strongly encouraged to work *independently*. That is the best way to understand & master the subject.
- Write the solutions on your own and honorably *acknowledge* the sources if any. `http://cse.iitk.ac.in/pages/AntiCheatingPolicy.html`
- Submit your solutions, before time, to your TA. Preferably, give the TA a printed copy of your LaTeXed or Word processed solution sheet.
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Problems marked '0 points' are for practice.

Question 1: *[3+4 points]* For an $m \times n$ matrix A, over a field, one can define the *rank* via columns or via rows. Prove that col-rk(A) = row-rk(A). Thus, we could talk about *the* rk(A).

Let B be another $m \times n$ matrix. Prove the *subadditivity* property: $\mathrm{rk}(A+B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$.

Question 2: *[10 points]* Let X be a nonnegative real-valued random variable with mean μ. Let $\delta \in (0,1)$. We are interested in estimating the probability of deviation of X from μ (multiplicatively). Show that:

(1) $\mathrm{Prob}[X > (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$, and

(2) $\mathrm{Prob}[X < (1-\delta)\mu] < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}$.

Question 3: *[6 points]* Let $n = d^2$ and consider the n-variate permanent polynomial, per_d, over $\mathbb{F}_q$. Prove that $\text{Prob}_{\alpha \in \mathbb{F}_q^n}[\ \text{per}_d(\alpha) \neq 0\ ] \geq \frac{1}{4}$.

Question 4: *[9+4+4 points]* We have extensively used the binomial estimates. They originate from Stirling's approximation. Prove that for all $n \in \mathbb{N}$,

$$\sqrt{2\pi} \ \leq\ \frac{n!}{n^{n+0.5}e^{-n}} \ \leq\ e\,.$$

Using this prove that:

(1) For $f + g = o(h)$, $\ln \frac{(h+f)!}{(h-g)!} \ =\ (f+g)\ln h \ \pm\ O\left(\frac{(f+g)^2}{h}\right)$,

(2) For constants $\alpha \geq \beta > 0$, $\ln \binom{\alpha n}{\beta n} \ =\ \alpha n \cdot H(\beta/\alpha) - O(\ln n)$.

Question 5: *[10 points]* Prove that the homogeneous-$\Sigma\Pi\Sigma$ complexity of $\det_d$ is $2^{\Omega(d)}$ (over any field).

Question 6: *[0 points]* What is the formula complexity of determinant?

Question 7: *[0 points]* Show that width-two ABP is an incomplete model.

Question 8: *[0 points]* Over constant-size $\mathbb{F}_q$, show that the d^2-variate polynomial $\text{sym}_{\leq d}$ requires depth-3 circuits of size $d^{\Omega_q(d)}$.

Question 9: *[0 points]* Complete the missing details in the exponential lower bound proof we did in the class; to express determinant as a multilinear depth-3 circuit.

Question 10: *[0 points]* Read about the 2021 breakthrough that proved super-polynomial lower bounds for constant-depth circuits. It uses the partial derivative measure, as done in the class. How do they *set-multilinearize* the constant-depth circuit?

Question 11: *[0 points]* For all the lower-bound measures $\mu(\cdot)$ that you've learnt, estimate $\mu(f)$ for a polynomial f with *random* coefficients in the base field.

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