CS748 - ARITHMETIC CIRCUIT COMPLEXITY
NITIN SAXENA

ASSIGNMENT 2

POINTS: 50

DATE GIVEN: 23-AUG-2024 DUE: 13-SEP-2024

Rules:

- You are strongly encouraged to work *independently*. That is the best way to understand & master the subject.
- Write the solutions on your own and honorably *acknowledge* the sources if any. `http://cse.iitk.ac.in/pages/AntiCheatingPolicy.html`
- Submit your solutions, before time, to your Tutor. Preferably, give the Tutor a printed copy of your LaTeXed or Word processed solution sheet.
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Problems marked '0 points' are for practice.

Question 1: *[7 points]* Recall the proof of $O(\log d)$-depth reduction, of size-s degree-d circuits, done in the class. Fill in the details showing that it actually gives a poly(sd)-time randomized algorithm to reduce depth of a given circuit.

Question 2: *[5+5 points]* Given a circuit C of size s and depth d, show that the circuit D computing all the first-order partial derivatives of C *has depth* $O(d)$, and size $O(s)$. [The latter was covered in the class.]

What is the best that you can say if one wanted to compute δ-order partial derivatives of C, for a given $\delta > 1$?

Question 3: *[13 points]* Recall the proof of "det $\in$ VP" that we saw, in the class, using Newton's identities. It was claimed that a linear system of n equations, that corresponds to a symbolic *triangular* matrix, can be solved by $O(\log n)$-depth poly(n)-size arithmetic circuit. Prove this claim.

Question 4: *[7 points]* Show that, over an integral domain R, the matrix $(\alpha_i^j)_{i,j\in[n]}$ is full-rank iff α_i's are distinct in R.

Question 5: *[9+4 points]* Over a field $\mathbb{F}$, show that the monomial $x_1 \dots x_n$ has a $\Sigma \wedge \Sigma$ expression iff $n! \neq 0$ in $\mathbb{F}$.

Question 6: *[0 points]* Recall the proof of $O(\log d)$-depth reduction, of size-s degree-d circuits, done in the class. Could the final size bound be made poly$(s \log d)$ in place of poly(sd)?

Question 7: *[0 points]* Consider a matrix $A \in R^{n\times n}$, where R is a possibly *non*-commutative ring. How will you define the determinant polynomial of A?

How fast can you compute it?

Question 8: *[0 points]* What is the smallest depth-3 circuit computing the $n \times n$ symbolic determinant? permanent?

Question 9: *[0 points]* Read about the 2021 breakthrough that proved exponential lower bounds for depth-3 circuits. What does it tell about determinant and permanent?

Question 10: *[0 points]* When we efficiently transform a formula to a width-3 (resp. constant-width) ABP $A_1 \cdots A_d$, can we ensure that the matrix product is *commutative*?

Question 11: *[0 points]* Over $\mathbb{F}_2$, can we efficiently transform a formula to a width-2 (resp. width-3) ABP? What if we allow the *approximative* computational analogues?

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