Homogeneous depth-4

- Homogeneity is a restriction for constant-depth circuits.

  (Not so for general circuits.)

- If a homogeneous $\Sigma\Pi^a\Sigma\Pi^b$ computes a degree d polynomial f, then we get the degree restriction $a, b \leq d$.

  Can this be used in shifted partials?

<u>Defn</u>: In a <u>homogeneous</u> depth-4 circuit $f(x_1, \ldots, x_n) = \sum_{i \in [s]} Q_{i1} \cdots Q_{ia_i}$, each Q_{ij} is a homogeneous sparse poly.

& $\sum_{j \in [a_i]} \deg Q_{ij} = \deg f$, $\forall i \in [s]$.

($\Rightarrow f$ is homogeneous too.)

- In homogeneous $\Sigma\Pi^a\Sigma\Pi^b$, b can be as high as the degree d of a polynomial f.

  So, we need to utilize the sparsity of the Q_{ij}'s.

- We will show, using random restrictions, that Q_{ij}'s can be "reduced" to a sum of $\sqrt{d}$-support-monomials.
  low-support

<u>Lemma:</u> Let f be an n-variate d-deg polynomial computable by a size $s \le n^{c\sqrt{d}}$ (constant $c>0$) homogeneous depth-4 C. Let ρ be a random restriction that sets each variable to 0 with probability $1-n^{-2c}$.

Then, with prob $\geq 1-\frac{1}{s}$, the polynomial $\rho(f)$ is computable by a homogeneous depth-4 C' with <u>bottom support</u> $\le \sqrt{d}$ & size $\le s$.

<u>Proof:</u>

- Among all Q_{ij} consider the monomials $\{m_1, \ldots, m_r\}$ that have support $>\sqrt{d}$. Clearly, $r \le s$.

$$\forall i \in [r], \ \Pr[\rho(m_i) \neq 0] < (n^{-2c})^{\sqrt{d}}$$

$$\Rightarrow \Pr[\exists i, \rho(m_i) \neq 0] < r n^{-2c\sqrt{d}} \le 1/s .$$

$\Rightarrow$ With prob $> 1-\frac{1}{s}$ all the large support monomials vanish. $\square$

- Now, we need to find a measure that is "small" for such $\Sigma\Pi\Sigma\Pi$.

Since we will prove a lower bound for a multilinear f, we can pick a measure that ignores the non-multilinear monomials.

Defn: For any $k, \ell \in \mathbb{N}$ & polynomial $f(\bar{x})$, define projected shifted partials $PSP_{k,\ell}(f)$ as the $\mathbb{F}$-span of the set of polynomials:

$$\{ mult(m_1 \cdot \partial_{m_2} f) \mid \deg m_1 = \ell, \deg m_2 = k \ \& \ m_1, m_2 \text{ are multilinear monomials} \}$$

where $mult(\cdot)$ refers to the projection to the multilinear part (eg. remainder modulo $\langle x_1^2, \dots, x_n^2 \rangle$).

The measure $\Gamma_{k,\ell}^{PSP}(f)$ is the dimension of $PSP_{k,\ell}(f)$.

Lemma 1 (Upper bd.): Let f be an n-variate d-degree polynomial computed by a homogeneous $\Sigma\Pi\Sigma\Pi$ of bottom-support $\leq r$ & size $\leq s$. Then, for any k, ℓ with $\ell + 4rk \leq \frac{n}{2}$ we have

$$\Gamma^{PSP}_{k,\ell}(f) \leq s \cdot \binom{d/r + k}{k} \cdot \binom{n}{\ell + 4rk}.$$

Proof:

- Consider a product gate $Q_{i1} \cdots Q_{ia}$.
- We could assume that the individual deg of any variable in Q_{ij} is ≤ 2.

  Otherwise, there is a monomial say x_1^3 which can never contribute to the polynomials $\mathrm{mult}(m_1 \partial_{m_2} f)$, as

  multilinear $m_2 \Rightarrow \partial_{m_2}(x_1^3)$ is non-multilr.

- Also, by multiplying out the Q_{ij}'s if needed, we can assume that $\deg Q_{ij} \in [r, 4r]$.
- Thus, we reduce to the case of $\Sigma\Pi^a\Sigma\Pi^b$, $a \leq d/r$ & $b \leq 4r$.

- Further, by using the multilinearity restrictions in the definition of $PSP_{k,\ell}(f)$, we get the upper bound. $\square$

- The lower bound of the measure is trickier.

  Because to get a result for a polynomial f one has to prove a measure lower bound for the various projections of f (under random restrictions ρ).

- Currently, such results are known for two types of polynomials:

Defn: • [Iterated matrix multiplication polynomial]

$IMM_{n,d}(\bar{x}) := (M_1 \cdots M_d)_{1,1}$

n^2d-variate
d-degree

where, $M_k = (x_{k,ij} \mid i,j \in [n])$ for $k \in [d]$.

• [Nisan-Wigderson polynomial] Let $\mathbb{F}_m$ be the finite field with m elements (identified with the elements $1,2,\dots,m$).

$\forall\, 0\le k\le n$, $NW_{n,m,k}(x_{11},\dots,x_{nm}) :=$

nm-variate
n-degree

$$\sum_{\substack{p(t)\in\mathbb{F}_m[t]\\ \deg p\le k}} x_{1,p(1)} \cdots\cdots x_{n,p(n)}$$

▷ $IMM_{n,d} \in VP$. (<u>OPEN</u>: $NW_{n,m,k} \in VP$?) $\in VNP$.

<u>Thm</u> [KLSS'14]: Over char zero, the homogeneous depth4 complexity of $NW_{d,d^3,d/3}$ is $d^{\Omega(\sqrt{d})}$.

<u>Thm</u> [KS'14]: The above holds for all $\mathbb{F}$. Further, $IMM_{n,d}$ has homogeneous depth4 complexity $d^{\Omega(\sqrt{d})}$.

– Proofs are left as reading exercises (from [Saptharishi '16]).