an $(n-k)$-minor of $\det_n$.

The leading monomial of this minor is merely the product of the variables in its principal diagonal.

leading monomial $\quad$ nonzero

$\Rightarrow LM(\partial_{\bar{p}} \det_n) = x_{i_1 j_1} \cdots\cdots x_{i_{n-k} j_{n-k}}$
where $i_1 < \cdots < i_{n-k}$ & $j_1 < \cdots < j_{n-k}$.

- Let us call such indices an $(n-k)$-increasing sequence in $[n] \times [n]$.

$\triangleright$ They are in bijection with $(n-k)$-minors.

$\Rightarrow \Gamma_{k,\ell}(\det_n) \geq \#$ monomials of $\deg \leq (n+\ell-k)$ that contain an $(n-k)$-increasing seq.

- To lower bound RHS we consider:

Defn: Let $D_2 := \{x_{11}, x_{22}, \ldots, x_{nn}\} \cup \{x_{12}, x_{23}, \ldots, x_{n-1,n}\}$ be the diagonal & the one above.

For monomial m define its

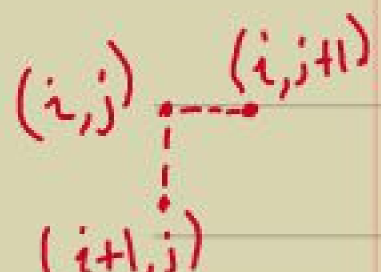
canonical increasing seq. $\chi(m)$ as the $(n-k)$-increasing seq. in m that is entirely contained in D_2 (& highest wrt $>$).

If the latter does not exist then define $\chi(m) := \phi$.

▷ Let S be an $(n-k)$-increasing seq. entirely contained in D_2 and m_S be its product. There are $\geqslant 2(n-k)-1$ variables in D_2 s.t. any monomial m in them satisfies:

$$\chi(m \cdot m_S) = \chi(m_S).$$

Proof:



- Note that for $(i,j) \neq (n,n)$, x_{ij} has a companion in D_2 of the type $x_{i+1,j}$ or $x_{i,j+1}$.
- Clearly, the variables in m_S, or their companions, do not alter $\chi(\cdot)$ when multiplied to m_S.

$\square$

▷ # $(n-k)$-increasing sequences, contained in D_2, is $\binom{n+k}{2k}$.

<u>Proof:</u> • We want to pick $(n-k)$ elements from

$x_{11}\ x_{12}\ x_{22}\ x_{23} \cdots\cdots x_{n-1,n}\ x_{nn}$

in a way that no two adjacent elements are picked.

• Consider the remaining $(2n-1) - (n-k) = n+k-1$ elements.

• Associate them with a string of $(n+k-1)$ 1's.

or at the two ends • We want to choose $(n-k)$ places in the <u>middle</u> of these 1's.

$$\Rightarrow \#\text{such choices} = \binom{(n+k-1)+1}{n-k}$$

$$= \binom{n+k}{n-k}.$$

□

$x(\cdot)$ • Note that this type of $(n-k)$-increasing sequence does not change if we multiply by $|X \setminus D_2| = (n^2-2n+1)$ many variables.

Moreover, we can multiply by at least $2(n-k)-1$ variables in D_2 without changing $x(\cdot)$.

Note: $mm_S = m'm_{S'}$ $\Rightarrow x(mm_S) = x(m'm_{S'})$ $\Leftrightarrow S = S'$.

$\Rightarrow$ We get the following lower bound on the number of distinct leading monomials in $\{x^{\bar\alpha} \cdot \partial_{\bar\beta} \det_n \mid |\bar\alpha| \le \ell, |\bar\beta| = k\}$:

$$\binom{n+k}{2k} \cdot \binom{n^2-2n+1+2(n-k)-1+\ell}{\ell}$$

$$= \binom{n+k}{2k} \cdot \binom{n^2-2k+\ell}{\ell} .$$

$\square$

- Now we have upper bounded $T_{k,\ell}$ for $\Sigma^s \Pi^a \Sigma \Pi^b$ & lower bounded for $\det_n$.

It is time to compare the two.

c is a constant

- For the applications $a = cn/b$ is of interest.
- For technical reasons, we use $k = \varepsilon n/b$ & $\ell = n^2 b$ (small enough constant $\varepsilon > 0$).

- By the two lemmas we get:

$$s \geqslant \binom{n+k}{2k} \cdot \binom{n^2-2k+\ell}{\ell} \Big/ \binom{cn/b+k}{k} \cdot \binom{n^2+(b-1)k+\ell}{n^2}.$$

<u>Claim 1:</u> $\ln \binom{n+k}{2k} = 2\varepsilon \frac{n}{b}\left(\ln \frac{b}{2\varepsilon}+1\right) \pm O(n/b^2)$.

<u>Claim 2:</u> $\ln \binom{n^2-2k+\ell}{\ell} \Big/ \binom{n^2+(b-1)k+\ell}{n^2} = -2\varepsilon\frac{n}{b}\left(\ln b+\frac{1}{2}\right) \pm O(1)$

<u>Claim 3:</u> $\ln \binom{cn/b+k}{k} = (c+\varepsilon)\cdot\frac{n}{b}\cdot H_e\left(\frac{\varepsilon}{c+\varepsilon}\right) - O(\ln n)$.

- These claims, after some calculations, imply:

$$\ln s \geqslant -\varepsilon\cdot\ln(4\varepsilon(c+\varepsilon))\cdot n/b \pm O(n/b^2)$$
$$= \Omega(n/b), \text{ for small } \varepsilon.$$

- The claims could be proved using the following binomial estimates:

$\ln \frac{(h+f)!}{(h-g)!} = (f+g)\ln h \pm O\left(\frac{(f+g)^2}{h}\right)$, if $f+g = o(h)$,

& $\ln \binom{\alpha n}{\beta n} = \alpha n \cdot H_e(\beta/\alpha) - O(\ln n)$,

for constants $\alpha \geq \beta > 0$.

- The proofs are left as exercises.

- This completes the proof of:

<u>Theorem</u> [GKKS'14]: Any $\Sigma^{s}\Pi^{O(n/b)}\Sigma\Pi^{b}$ circuit computing $\det_n$ or per_n requires $s = \exp(\Omega(n/b))$.

- For $b = \sqrt{n}$, this shows that the depth reduction to depth-4 is <u>almost optimal</u>.

(∵ $\det_n$ has such a circuit of size $n^{O(\sqrt{n})}$.)

This was further clarified by:

<u>Thm</u> [Fournier, Limaye, Malod, Srinivasan '14]: For a small $\delta > 0$ & $d \leq n^{\delta}$, any $\Sigma^{s}\Pi^{O(\sqrt{d})}\Sigma\Pi^{\sqrt{d}}$ circuit computing $\mathrm{IMM}_{n,d}$ has $s = n^{\Omega(\sqrt{d})}$.

optimal