Degree-restricted depth-4

- Recall that a depth-4 circuit of the type $\Sigma\Pi^a\Sigma\Pi^b$ has the form $f = \sum_{i\in[s]} Q_{i1}\cdots Q_{ia}$ in $\mathbb{F}[x_1,..,x_n]$, where $\deg(Q_{ij}) \le b$.

- We know that a size-s deg-d f has a $\Sigma\Pi^{O(\sqrt{d})}\Sigma\Pi^{\sqrt{d}}$ circuit of size $s^{O(\sqrt{d})}$.

$\omega(\cdot)$ is small omega

Conversely, if f requires $s^{\omega(\sqrt{d})}$ size $\Sigma\Pi^{O(\sqrt{d})}\Sigma\Pi^{\sqrt{d}}$ circuits then it requires $s^{\omega(1)}$ size arbitrary circuits.

- To study this model (Kayal '12) modified the partial derivative based measures.

Definition: Let $\partial^{=k}(f)$ be the set of order-k partial derivatives of f & $x^{\le \ell}$ be the monomials of deg $\le \ell$.

The shifted partials of f, denoted

by $\langle \partial^{=k} f \rangle_{\leq \ell}$, is the $\mathbb{F}$-vector space spanned by $\{ x^{\bar{e}} \cdot \partial_{\bar{d}} f \mid |\bar{e}| \leq \ell, |\bar{d}| = k \}$.

The dimension of shifted partials is denoted by $\Gamma_{k,\ell}(f)$.

- The matrix, wrt f, whose rank we are interested in is:

$$\begin{array}{cc} & m \\ x^{\bar{\alpha}} \cdot \partial_{\bar{\beta}} & \left. \underbrace{\begin{pmatrix} \mathrm{coef}(m)(x^{\bar{\alpha}} \cdot \partial_{\bar{\beta}} f) \end{pmatrix}}_{\text{n-var. monomials of deg} \leq \ell + d - k} \right\} x^{\leq \ell} \partial^{=k} \end{array}$$

▷ Clearly, $\Gamma_{k,\ell}$ is sub-additive.

Pf: Derivation is an $\mathbb{F}$-linear operation. □

Lemma 1: Let f be an n-variate computed by a $\Sigma^{s} \Pi^{a} \Sigma \Pi^{b}$ circuit. Then,

$$\Gamma_{k,\ell}(f) \leq s \cdot \binom{a+k}{k} \cdot \binom{n + (b-1)k + \ell}{n}.$$

Proof:

- By subadditivity, it suffices to

Consider a product gate $f = Q_1 \cdots Q_a$ with $\deg Q_i \le b$.

- For a $\bar\beta$, $|\bar\beta| = k$, $\partial_{\bar\beta} := \partial_{x^{\bar\beta}}(Q_1 \cdots Q_a)$ can be expanded using the product rule of derivation.
- The number of summands there is $\le \binom{a+k}{k}$.

in this summand we can ignore $Q_{k+1} \cdots Q_a$

- Now, by subadditivity, we reduce to cases of the type: $\partial_1 Q_1 \cdots \partial_k Q_k$.

$\Rightarrow$ after monomial multiplication we have products like $x^{\bar\alpha} \cdot \prod_{i \in [k]} \partial_i Q_i$, $|\bar\alpha| \le \ell$.

- The number of monomials here is $\le$
$$\binom{n+\deg}{n} \le \binom{n+(b-1)k+\ell}{n}.$$

$$\Rightarrow \quad T_{k,\ell}(f) \le \binom{a+k}{k} \cdot \binom{n+(b-1)k+\ell}{n}.$$

compare this with $\binom{n+k}{k} \cdot \binom{n+\ell}{\ell}$

$\square$

wrt RHS

- Thus, we want an f with a "large" $T_{k,\ell}(f)$ for some parameters k & ℓ.

- We will now lower bound $T_{k,\ell}$ for $\det_n$ (& similarly per_n).

<u>Lemma 2</u>: [Gupta, Kamath, Kayal, Saptharishi '14]:

$$T_{k,\ell}(\det_n) \geq \binom{n+k}{2k} \cdot \binom{n^2-2k+\ell}{\ell}.$$

<u>Proof</u>:

- Say $\det_n$ has variables x_{ij}, $i,j \in [n]$.
- Let us fix a monomial ordering as:

$$x_{11} > x_{12} > \dots > x_{1n} > \dots > x_{n1} > \dots > x_{nn}.$$

- Under this ordering we want to estimate the number of <u>leading</u> monomials in the polynomials in the set

$$\{ x^{\bar{\alpha}} \cdot \partial_{\bar{\beta}} \det_n \mid |\bar{\alpha}| \leq \ell, |\bar{\beta}| = k \}.$$

- Clearly, that estimate is a lower bound on $T_{k,\ell}(\det_n)$.

- Note that $\partial_{\bar{\beta}} \det_n$ is either zero or an $(n-k)$-minor of $\det_n$.

The leading monomial of this minor is merely the product of the variables in its principal diagonal.

leading monomial nonzero

$\Rightarrow LM(\partial_\rho \det_n) = x_{i_1 j_1} \cdots\cdots x_{i_{n-k} j_{n-k}}$ where $i_1 < \cdots < i_{n-k}$ & $j_1 < \cdots < j_{n-k}$.

- Let us call such indices an (n-k)-increasing sequence in $[n] \times [n]$.

$\triangleright$ They are in bijection with $(n-k)$-minors.

$\Rightarrow \Gamma_{k,\ell}(\det_n) \geq$ # monomials of deg-$(n+\ell-k)$ that contain an $(n-k)$-increasing seq.

- To lower bound RHS we consider:

Defn: Let $D_2 := \{x_{11}, x_{22}, \ldots, x_{nn}\} \cup \{x_{12}, x_{23}, \ldots, x_{n-1,n}\}$ be the diagonal & the one above.

For monomial m define its