- Now we need to study the effect of a random partitioning on a $t$-product.

<u>Lemma:</u> Let $f(X)$ be $n$-variate & computable by a size-$s$ multilinear depth-$\Delta$ formula. If $X = Y \sqcup Z$, $|Y| = |Z| = n/2$, is random then with probability $1 - s \cdot \exp(-n^{\Omega(1/\Delta)})$:

the constants in $\Omega$ behave well

$$\Gamma_{Y,Z}(f) = s \cdot 2^{n/2} \cdot \exp(-n^{\Omega(1/\Delta)}).$$

<u>Proof:</u>

- By the previous lemma, write $f = g_0 + \sum_{i=1}^{s} g_i$ where $\deg g_0 \le n/100$ & $g_1, \dots, g_s$ are multilinear $t$-products.
- Note that $g_0$'s sparsity can be at most

$$\sum_{i \le n/100} \binom{n}{i} = 2^{H_2(1/100) \cdot n - O(\lg n)} < 2^{n/10}.$$

$$\Rightarrow \Gamma_{Y,Z}(g_0) < 2^{n/10} \quad \text{(sub-additivity)}.$$

- All that remains is to bound $\Gamma_{Y,Z}(g_1)$ for a random partition $X = Y \sqcup Z$.

- Let $g = h_1 \cdots h_t$, $h_i \in \mathbb{F}[X_i]$, be a $t$-product for $X = \sqcup X_i$.

Let $Y_i := X_i \cap Y$ & $Z_i := X_i \cap Z$.

- Let $d_i := |\#Y_i - \#Z_i|/2$ be the <u>imbalance</u> between $Y_i, Z_i$ in $h_i$.

$X_i$ is called <u>$k$-imbalanced</u> if $d_i \geq k$.

Let $b_i := (\#Y_i + \#Z_i)/2 = \#X_i/2$.

- We have $T_{Y,Z}(g) = \prod_i T_{Y,Z}(h_i) \leq \prod_i 2^{\min(|Y_i|, |Z_i|)}$

$$= \prod_i 2^{b_i - d_i} = 2^{|X|/2} / \prod_i 2^{d_i}.$$

$\Rightarrow$ it suffices to show that one of the $X_i$'s is imbalanced (i.e. $d_i$ is large).

- We need to estimate $|Y_i|$ on choosing a random $Y \in \binom{[n]}{n/2}$.
- The relevant probability is that of the <u>hypergeometric distribution</u>:

**Claim:** For a fixed set $A \in \binom{[n]}{a}$, $k \leq a \leq 2n/3$,

$$\Pr_{R \in \binom{[n]}{n/2}} [|R \cap A| = k] = O(1/\sqrt{a}).$$

**Proof:**
- $\Pr_R[|R \cap A| = k] = \binom{a}{k} \cdot \binom{n-a}{n/2-k} / \binom{n}{n/2}$
- Call it $P(k)$.
- $P(k+1) > P(k)$ iff

$$(a-k)\left(\frac{n}{2}-k\right) > (k+1)\left(\frac{n}{2}-a+k+1\right) \text{ iff}$$

$$\frac{an}{2} - k\left(a+\frac{n}{2}\right) > \left(\frac{n}{2}-a+1\right) + k\left(\frac{n}{2}-a+2\right)$$

iff $k < \frac{a-1}{2}$.

- Thus, $P(k) \leq \binom{a}{\frac{a-1}{2}} \cdot \binom{n-a}{\frac{n}{2}-\frac{a-1}{2}} / \binom{n}{n/2}$

(Stirling's approx.) $$= O\left(\sqrt{\frac{n}{a(n-a)}}\right) = O\left(\frac{1}{\sqrt{a}}\right). \quad \square$$

(k-balanced)
- Let $\mathcal{E}_i$ denote the event that $d_i < k$.
- We have $\Pr[\bigwedge_{i=1}^{t} \mathcal{E}_i]$ equal to

$$\Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 | \mathcal{E}_1] \cdot \Pr[\mathcal{E}_3 | \mathcal{E}_1 \wedge \mathcal{E}_2] \cdots\cdots.$$

- $\Pr[\mathcal{E}_1] = \Pr_Y[|Y \cap X_1| \in [b_1 - k, b_1 + k]]$

which the above claim estimates as:

$$k \cdot O(1/\sqrt{b_1}) \quad \text{(assuming } k \leq b_1/2\text{)}.$$

- Consider the event $\mathcal{E}_i$ given $\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}$. Since $X_1, \ldots, X_{i-1}$ have been partitioned in a fairly balanced way ($\forall j \in [i-1], d_j < k$), we deduce that $|Y \cap (X_1 \cup \ldots \cup X_{i-1})^c|$

$$\begin{aligned} &= |Y \cap X| - |Y \cap (X_1 \cup \ldots \cup X_{i-1})| \\ &< n/2 - (b_1 - k + \ldots + b_{i-1} - k) \\ &= (n/2 - b_1 - \ldots - b_{i-1}) + (i-1)k \end{aligned}$$

$\Rightarrow$ The partition of $X' := (X_1 \sqcup \ldots \sqcup X_{i-1})^c$ by $Y \sqcup Z$ is $(ik)$-balanced.

- So, assuming $ik \ll n$, we can redo the calculation in the above claim & still get $\Pr[\mathcal{E}_i | \mathcal{E}_1 \wedge \ldots \wedge \mathcal{E}_{i-1}] = k \cdot O(1/\sqrt{b_i})$.

$$\Rightarrow \Pr\Big[\bigwedge_{i \in [t]} \mathcal{E}_i\Big] = O\big(k^t / \sqrt{b_1 \cdots b_t}\big)$$

$$\Rightarrow \Pr_Y\big[T_{Y,Z}(g) > 2^{|X|/2} \cdot 2^{-k}\big] = O\big(k^t / \sqrt{b_1 \cdots b_t}\big)$$

• In particular, on fixing $k \le t^{1/3}$, we get: $\Pr_Y[T_{Y,Z}(g) > 2^{n/2} \cdot 2^{-k}] = O(\prod_{i=1}^{t} t^{-1/6})$
$$= O(t^{-t/6}) = \exp(-n^{\Omega(1/\Delta)}).$$

$\Omega$ independent of $\varepsilon$ if $\varepsilon \le 1/3$

$$\Rightarrow \Pr_Y[T_{Y,Z}(f) > s \cdot 2^{n/2-k} =: s 2^{n/2} \cdot 2^{-t^{\varepsilon}}]$$
$$= s \cdot \exp(-n^{\Omega(1/\Delta)}).$$

$\square$

– Thus, there is an $\varepsilon \in (0, \frac{1}{3}]$ such that if $s \le \exp(t^{\varepsilon})$ then $\Pr[T_{Y,Z}(f) = 2^{n/2}] < 1/10$.

$\Rightarrow$ $f(X)$ could compute $\det(X)$ only if $s > 2^{t^{\varepsilon}} = \exp(n^{\Omega(1/\Delta)})$.

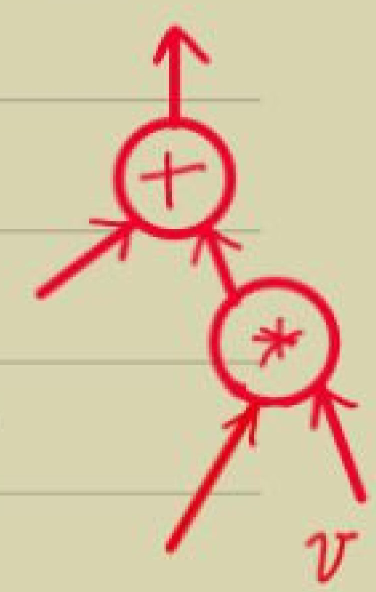
– This finishes (Raz, Yehudayoff '09) proof for $\det_n$ or $\mathrm{per}_n$ against constant-depth multilinear model.

- We can also say something for multilinear formulas using the probability calculation seen above.

- The multilinear products of interest there are:

<u>Defn:</u> Multilinear $f = \prod_{i=1}^{t} g_i$, with partition $X = \bigsqcup_{i \in [t]} X_i$, is called a <u>log-product</u> if for all $i$, $|X|/3^i \leq |X_i| \leq 2 \cdot |X|/3^i$ and $|X_t| = 1$.

<u>Lemma:</u> Any size-$s$ multilinear formula $\phi$ can be written as a sum of $(s+1)$ log-products.




<u>Proof:</u>
- Let $|X| > 2$ & $\phi$ compute $f$.
- Let $v$ be a node in $\phi$ that depends on variables $X_v$ such that

assume fanin 2 in $\phi$ → $|X|/3 \leq |X_v| \leq 2 \cdot |X|/3$

- By the formula properties, we have

$$f = \phi_v \cdot g + \phi_{v=0}$$

for some $g \in \mathbb{F}[X \setminus X_v]$.

- Note that $|X|/3 \leq |X \setminus X_v| \leq 2 \cdot |X|/3$.
- Moreover, since $g$ has size $< s$, we can use induction & write it as a sum of $\leq$ size$(g)+1$ log-products.

Similarly, for $\phi_{v=0}$.

$\Rightarrow$ $f$ is a sum of $(s+1)$ log-products.

$\square$

– Now, we can estimate $\Gamma_{Y,Z}(h_1 \cdots h_t)$ for a log-product $h_1 \cdots h_t$, $t = O(\lg n)$.

Note that around $\frac{1}{2} \lg n$ many of these $h_i$'s do depend on at least $\sqrt{n}$ many variables each.

$\Rightarrow$ On doing the probability calculation we will get that $\Gamma_{Y,Z}(h_1 \cdots h_t)$ is high with prob. smaller than $n^{-\Omega(\lg n)}$.

(Raz '09) $\Rightarrow$ $\det_n$ or $\mathrm{per}_n$ requires $n^{\Omega(\lg n)}$ size!