$\Rightarrow \Gamma_{Y,Z}(f) \leq 2^{d-d/16} \leq 2^{n/2} \cdot 2^{-n/32}$.

(as $n/3 \leq d \leq n/2$) $\square$

Det_n & Per_n have high $\Gamma_{Y,Z}(\cdot)$

- Note that $\det_n$ has n^2 variables.
  We will first reduce its variables to a random X, $|X| = 2m := 2 \cdot \frac{\sqrt{n}}{5}$, & then use a random partition $X = Y \sqcup Z$.

Theorem (Lower bound) [Raz'09]: With probability $\geq 1/2$, a random restriction σ of $\{x_{11}, \ldots, x_{nn}\}$ to $X = Y \sqcup Z$, $|Y| = |Z| = m := \lfloor \sqrt{n}/5 \rfloor$, yields $\Gamma_{Y,Z}(\sigma \circ \det_n) = 2^m$.

Proof:

- The map σ will fix $n^2 - 2m$ variables to $\mathbb{F}$ values, in a certain way.
- The remaining $2m$ variables are X.
- Let us compute the probability that

these variables <u>do not</u> share the same row or column.

- $\Pr_{\sigma}$ [X have diff. rows/cols]

$$= \frac{n^2}{n^2}\cdot\frac{(n-1)^2}{n^2}\cdot\ldots\cdot\frac{(n-2m+1)^2}{n^2} = \left(1-\frac{1}{n}\right)^2\cdots\left(1-\frac{2m-1}{n}\right)^2$$

$$> \left(1-\sum_{i=1}^{2m-1}\frac{i}{n}\right)^2 > 1-\frac{2m\cdot(2m-1)}{n} > 1-\frac{4}{25}.$$

- For such a σ the determinant shares properties with:

$$\begin{vmatrix} y_1 & 1 & & & & & \\ 1 & z_1 & & & & 0 & \\ & & \ddots & & & & \\ & & & y_m & 1 & & \\ & 0 & & 1 & z_m & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{vmatrix}$$

$$= (y_1 z_1 - 1)\cdots(y_m z_m - 1) =: D_m.$$

error prob $\leq \frac{1}{3}$ $\rightarrow$ if $|\mathbb{F}| \geq 3$.

- In particular, we have whp, for a random σ: $T_{y,z}(\sigma(\det_n)) = T_{y,z}(D_m)$. (Exercise!)

- Clearly, $T_{y,z}(D_m) = 2^m$. (Multiplicativity)

$\Rightarrow \Pr_{\sigma}[T_{y,z}(\sigma \circ \det_n) = 2^m] > 1 - \frac{4}{25} - \frac{1}{3}$

$> 1/2$. $\square$

- Finally, we deduce an exponential lower bound against multilinear depth-3.

Corollary: $\det_n$ or per_n require $2^{\Omega(\sqrt{n})}$ size multilinear depth-3 circuits.

Proof:

- Suppose $\det_n = C(\overline{x})$, for a multilinear $\Sigma^s \Pi \Sigma$ circuit C.
- We apply, as before, a random variable reduction σ both sides

$\Rightarrow \sigma \circ \det_n = \sigma \circ C$ (2m-variate now).

- The two theorems imply that

$2^m \leq s \cdot 2^{2m/2 - 2m/32}$

$$\Rightarrow \quad s \geq 2^{m/16} = 2^{\sqrt{n}/80}. \qquad \square$$

- Note that this almost matches the best depth-3 complexity of $\det_n$.

Exercise: The same argument holds for per_n.

Generalizing to constant-depth multilr

- (Raz, Yehudayoff '09) generalized the above ideas to get a result for multilinear depth-Δ circuits.

- Here, instead of a product of linear polynomials we work with the following:

Defn: A multilinear polynomial $f = g_1 \cdots g_t$ is called a t-product if:
each g_i depends on $\geq t$ variables.

Lemma: Let f be a multilinear n-variate d-degree polynomial that has a size-s multilinear (product) depth-Δ formula ϕ. Then, f can be written as a sum of: $\leq s$ multilinear t-products ($t = (n/100)^{1/2\Delta}$) & a multilinear polynomial of degree $\leq n/100$.

Proof:

- If $d \leq n/100$, then it is clear.
- Let $d > n/100$. Since ϕ is a formula of product-depth Δ, there is a product gate v of fanin $\geqslant (n/100)^{1/\Delta} =: t^2$.
- Let us expand the formula wrt this gate:

$$f = \phi_v + \phi_{v=0}$$

output at v

- As ϕ_v is a product of t^2 polynomials we can group them to see that ϕ_v is multilinear t-product.
- As $\phi_{v=0}$ is of smaller size, we can recurse. $\square$

- Now we need to study the effect of a random partitioning on a t-product.

<u>Lemma:</u> Let $f(X)$ be n-variate & computable by a size-s multilinear depth-Δ formula. If $X = Y \sqcup Z$, $|Y| = |Z| = n/2$, is random then with probability $1 - s \cdot \exp(-n^{\Omega(1/\Delta)})$:

$$\Gamma_{Y,Z}(f) = s \cdot 2^{n/2} \cdot \exp(-n^{\Omega(1/\Delta)}).$$

<u>Proof:</u>

- By the previous lemma, write $f = g_0 + \sum_{i=1}^{s} g_i$ where $\deg g_0 \le n/100$ & $g_1, \ldots, g_s$ are multilinear t-products.
- Note that g_0's sparsity can be at most

$$\sum_{i \le n/100} \binom{n}{i} = 2^{H_2(1/100) \cdot n - O(\lg n)} < 2^{n/10}.$$

$\Rightarrow \Gamma_{Y,Z}(g_0) < 2^{n/10}$ (sub-additivity).

- All that remains is to bound $\Gamma_{Y,Z}(g_1)$ for a random partition $X = Y \sqcup Z$.