$\Rightarrow \mathcal{E} := \bigcup_{rk(T)>\tau} \mathcal{E}_T$ has size $< s \cdot \varepsilon^{-\tau/8q} \cdot q^n$

& $A := \mathbb{F}_q^n \setminus \mathcal{E}$ zeroes out every function in $\partial^{=k} T$, for $T \in \{T_i \mid i \in [s], rk(T_i) > \tau\}$.

$\Rightarrow \Gamma_{k,A}(C)$ is contributed by only T_i's with $rk(T_i) < \tau$

$\Rightarrow \Gamma_{k,A}(C) < s \cdot q^{\tau}$. □

- Next, we understand the measure for $\det_d$ & per_d, $n := d^2$.

<u>Lemma 2</u> (Lower bound): For any $A \subseteq \mathbb{F}_q^n$ of size $(1-o(1))q^n$, we have $\Gamma_{k,A}(\det_d) = \binom{d}{k}^2$.

<u>Proof:</u> (from Saptharishi's survey)

- We consider the rank of the matrix $M_k(\det_d, A)$.
- An order-k derivative (partial) of $\det_d$ is, either zero, or an order-$(d-k)$ <u>minor</u>.

• Since $\det_d$ has $\binom{d}{d-k}^2$ many order-$(d-k)$ minors, it can be seen that the rank of $M_k(\det_d, \mathbb{F}_q^n) = \binom{d}{k}^2$.

[We can pick a point $\bar{\alpha} \in \mathbb{F}_q^n$ s.t. the column $\bar{\alpha}$ has exactly one nonzero entry in $M_k(\det_d, \mathbb{F}_q^n)$: a desired order-$(d-k)$ minor. Thus, we identify a "diagonal" matrix inside $M_k(\cdot,\cdot)$; lower bounding its rank.]

• However, $M_k(\det_d, A)$ has, possibly, many columns missing. How do we lower bound its rank?

<u>Idea</u>- We study arbitrary linear combinations of its rows.

<u>Claim 1</u>: Let $f(\bar{x})$ be a $\mathbb{F}_q$-linear combination of $r \times r$ minors of $X = (x_{ij})$. Then,

$$\Pr_{\bar{\alpha} \in \mathbb{F}_q^n} [f(\bar{\alpha}) \neq 0] \geq \frac{1}{4}.$$

- This claim immediately implies that the rows of $M_k(\det_d, A)$, corresponding to the minors, are linearly independent;

Since the #zeros, of a linear combination of minors of X, is $\leq \frac{3}{4}q^n$ & so $|A| - \frac{3}{4}q^n = \frac{1}{4}q^n - o(1)q^n > 0$.

$\Rightarrow$ We only need to prove Claim 1.
First, we prove a base case:

<u>Claim 2:</u> $\Pr_{\bar{\alpha} \in \mathbb{F}_q^n} [\det_d(\bar{\alpha}) \neq 0] \geq 1/4$.

<u>Proof:</u> - The number of <u>invertible</u> matrices in $\mathbb{F}_q^{d \times d}$ is $(q^d - 1) \cdot (q^d - q) \cdot \cdots \cdot (q^d - q^{d-1})$.

$\Rightarrow$
$$\Pr_{\bar{\alpha}} [\det_d(\bar{\alpha}) \neq 0] = \left(1 - \frac{1}{q}\right) \cdot \left(1 - \frac{1}{q^2}\right) \cdots \left(1 - \frac{1}{q^d}\right)$$
$$\geq \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2^2}\right) \cdots \left(1 - \frac{1}{2^d}\right) \geq \frac{1}{4}.$$

$\square$

<u>Exercise:</u> Prove Claim 2 for per_d.

Pf of Clm 1: • Let the linear combination of the $r\times r$ minors of $\det_d(X)$ be

$$f(\bar{x}) = \sum_{\text{row-1 in } M_i} c_i \cdot M_i + \sum_{\substack{\text{row-1 not} \\ \text{in } M_j}} c_j \cdot M_j .$$

• We now want to further expand each M_i by row-1 of X & rearrange the first part of $f(\bar{x})$ above:

$$f(\bar{x}) = \sum_{i\in[d]} x_{1i} \cdot M_i' + M'' .$$

Now M_i' are $\mathbb{F}_q$-linear combinations of certain order-$(r+1)$ minors of $\det_d(X)$. M'' is "free" of x_{1j} variables.

• Wlog we can assume that at least two distinct order-r minors participated in defining $f(\bar{x})$, and that at least one of the M_i' above is nonzero.

• We would like to pick a random $\bar{\alpha}$ by first picking the rows $\{2,..,d\}$ & picking the first row in the end (from $\mathbb{F}_q^d$).

- From this viewpoint it is clear that:

$$\text{LHS} = \Pr_{\bar{\alpha}}\left[\sum_{i=1}^{d} \alpha_{1i} \cdot M_i'\Big|_{\bar{\alpha}} + M''\Big|_{\bar{\alpha}} \neq 0 \right]$$

$$\geqslant \Pr_{\bar{\alpha}}\left[\sum_{i=1}^{d} \alpha_{1i} \cdot M_i'\Big|_{\bar{\alpha}} \neq 0 \right] \quad \text{(koutis' trick)},$$

- The latter involves only the minors that have row-1 of X.
- Repeating this, several times, we end up with the probability estimate for a single minor (as in Clm 2).

$$\Rightarrow \text{LHS} \geqslant 1/4.$$

$\square$

- As discussed before Clm 1 implies that $T_{k,A}(\det_d) = \binom{d}{k}^2$, finishing Lem 2.

$\square$

<u>Exercise</u>: Prove the same for per_d.

- Assuming that $\det_d$ has a depth-3 circuit, we compare the bounds in Lemmas 1 & 2: let $\tau = \alpha d$, $k = \tau/10q$,

$$\binom{d}{k}^2 = \Gamma_{k,A}(\det_d) < s \cdot q^{\alpha d}$$

[Stirling's approx. gives: $\lg \binom{n}{\varepsilon n} = H_2(\varepsilon) \cdot n - O(\lg n)$, where $H_2(\varepsilon) := -\varepsilon \lg \varepsilon - (1-\varepsilon) \lg (1-\varepsilon)$.]

[Eg. it follows that $\binom{n}{\varepsilon n} = 2^{\Omega_\varepsilon(n)}$.]

$$\Rightarrow \lg \binom{d}{k}^2 = \Omega\left(d \cdot H_2(k/d)\right) = \Omega\left(d \cdot H_2(\alpha/10q)\right)$$

$$\Rightarrow \lg s = \Omega\left(d H_2\left(\tfrac{\alpha}{10q}\right)\right) - \alpha d \lg q$$

$$\Rightarrow \lg s/d = \Omega\left(\frac{\alpha}{10q} \lg \frac{10q}{\alpha} + \left(1 - \frac{\alpha}{10q}\right) \lg \frac{10q}{10q - \alpha}\right) - \alpha \cdot \lg q$$

- Thus, there is some constant $c > 0$ s.t. it suffices to pick α satisfying

$$\lg \frac{10q}{\alpha} > cq \cdot \lg q$$

$\Leftrightarrow \alpha < 10q/q^{cq}$. Thus, $\tau = O(d/q^{cq-1})$.

- For constant q, τ makes sense & we get a lower bound on the top fanin:

$$\lg s = \Omega_q(d)$$

finishing the theorem. $\square$

- The lower bound can be improved by considering a sum of elementary symmetric polynomials on $n = d^2$ variables & $\deg \leq d$.

Define $\text{sym}_{\leq d} := \sum_{S \in \binom{[n]}{\leq d}} x_S$.

- It can be shown that the rank of the matrix $M_k(\text{sym}_{\leq d}, \mathbb{F}_q^n) \geq \binom{n}{d/2}$, for $k = d/2$.

- This gives $s = n^{\Omega_q(d)}$.