Width reduction in ABP

- We now explore the power of constant-width ABP.

Theorem [Ben-Or, Cleve '88]: Formulas & width-3 ABP are equivalent up to poly-size.

Proof:

- Let F be a formula of size-s.
  Wlog we can assume it to be of fanin two & depth $d = O(\lg s)$. [Brent's formula reduction]

- We intend to compute F by IMM of 3×3 matrices using bottom-up induction.
- Gate $E \in \mathbb{F} \cup \bar{x}$ can be computed as (base case): $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ E & 0 & 1 \end{pmatrix}$

  Idea: $(R_1, R_2, R_3) \mapsto (R_1 + R_3 \cdot E, R_2, R_3)$

- Gate $E = E_1 + E_2$ can be computed as:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ E_1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ E_2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ E & 0 & 1 \end{pmatrix}$$

- Gate $E = E_1 \cdot E_2$ computed as:

$$\begin{pmatrix} 1 & 0 & 0 \\ -E_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & E_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ E_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -E_1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -E_2 & 1 & 0 \\ 0 & E_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ E_2 & 1 & 0 \\ 0 & -E_1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ E_1 \cdot E_2 & 0 & 1 \end{pmatrix}.$$

- Note that $\begin{pmatrix} 1 & 0 & 0 \\ f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ refers to the step

$(R_1, R_2, R_3) \mapsto (R_1 + R_2 \cdot f, R_2, R_3)$. Thus, its IMM form mirrors that for $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f & 0 & 1 \end{pmatrix}$.

The latter we have constructed by induction.

$\Rightarrow$ The IMM for $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F & 0 & 1 \end{pmatrix}$ can be found by induction & is of size $\leq 4^d$ $= \mathrm{poly}(s)$.

▷ IMM for F is poly(s)-size & involves permuted triangular 3×3 matrices of det = 1.

- For the converse, let $A = A_1 \cdots A_s$ be a product of 3×3 symbolic matrices.

- Suppose we have size $T(s/2)$ formulas for $L = A_1 \cdots A_{s/2}$ & $A_{s/2+1} \cdots A_s = R$.

  As L & R are 3×3 matrices we can make 3 copies of each of their entries & get a formula for $L \cdot R$ of size:

  $T(s) = 6 \cdot T(s/2) + O(1)$

  $\Rightarrow \ T(s) = O(6^{\lg s}) = o(s^3)$.

▷ constant-width IMM A has poly(s)-size formula.

$\square$

▷ $O(1)$-depth ckt. $\leq$ formula $\equiv$ $O(1)$-width ABP

▷ quasipoly formula $\geq$ ABP $\leq$ low-deg circuit.

Width-2 Chasm

- We now show that even triangular 2×2 matrices give a strong ABP.

Theorem [Saha, Saptharishi, S. '09]: Let f be a $\Sigma^k\Pi^d\Sigma^{n+1}$ polynomial. There is a size $O(dk^2)$ width-2 ABP that computes $f\cdot L$, where L is a product of nonzero linear polynomials.

[The proof gives a poly(dkn)-time algorithm to compute a width-2 ABP with upper triangular matrices.
In this sense the ABP uses the minimum amount of non-commutativity.]

Proof:

- Let $f = \sum_{i=1}^{k} P_i$, where $P_i = \prod_{j=1}^{d} \ell_{ij}$ is a product of linear polynomials.

- Observe that P_i can be computed by

the length-d matrix product:

$$\begin{pmatrix} l_{i1} & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} l_{i,d-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l_{id} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} P_i' & P_i \\ 0 & 1 \end{pmatrix} \quad , \quad P_i' := P_i / l_{id} .$$

- Now we need to express their sum, i.e. $P_1 + P_2$.
- More generally, suppose we have IMM for multiples of g & h as:

$$\begin{pmatrix} L_1 & L_2 g \\ 0 & L_3 \end{pmatrix} \ \& \ \begin{pmatrix} M_1 & M_2 h \\ 0 & M_3 \end{pmatrix} .$$

- Can we express $g+h$ in a similar way?
- We can try scaling them by A, B as:

$$\begin{pmatrix} L_1 & L_2 g \\ 0 & L_3 \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} M_1 & M_2 h \\ 0 & M_3 \end{pmatrix} =$$

$$\begin{pmatrix} AL_1 M_1 & AL_1 M_2 h + BL_2 M_3 g \\ 0 & BL_3 M_3 \end{pmatrix}$$

- Clearly, this suggests picking $A = L_2 M_3$ & $B = L_1 M_2$ to get
$$\begin{pmatrix} L_1 L_2 M_1 M_3 & L_1 L_2 M_2 M_3 (g+h) \\ 0 & L_1 L_3 M_2 M_3 \end{pmatrix}$$
as an IMM.

- The size of the IMM has increased by at most a multiple of 4.

- We can apply this recursive step to $\sum_{i=1}^{k} P_i$ at most $\lg k + 1$ times.

$\Rightarrow$ We get an IMM of length $\leq d \cdot 4^{\lg k+1} = O(dk^2)$ for
$$\begin{pmatrix} L' & L \cdot f \\ 0 & L'' \end{pmatrix}$$
where, L is a product of $O(dk)$ linear polynomials (that appear in the circuit).

$\square$