- Thus, we get recurrences for the size & depth functions:

$$\mathrm{size}(s) \leq 4 \cdot \mathrm{size}(2s/3) + O(1),$$
$$\mathrm{depth}(s) \leq \mathrm{depth}(2s/3) + O(1).$$

$\Rightarrow$ $\mathrm{size}(s) = \mathrm{poly}(s)$ &
$\mathrm{depth}(s) = O(\lg s)$.

$\square$

- In a general circuit there will be more overlap between $C_{v=y}$, C_v & so the above argument does not work.

- However, a different argument will work – based on recursively reducing the <u>degree</u> as we walk down.

<u>Theorem</u> (Valiant, Skyum, Berkowitz, Rackoff '83]: Let $\deg = d$ polynomial f be computed by a size-s circuit C. Then, there is a

poly($snlgd$)-size, depth-$O(\lg d)$ circuit C' computing f.

[Moreover, given C there is a randomized poly(snd)-time algorithm to construct C'.]

Proof: (We use Saptharishi's (2016) exposition.)

- Wlog, we assume that C has fanin 2 & that for any gate v with left (resp. right) child v_L (resp. v_R), $\deg v_R \geq \deg v_L$.
  [We call C right-heavy.]
- By $[v]$ we will denote the polynomial computed at gate v.
  Also, $[v]$ will be a node in the new circuit C'.

Defn: For gates u, v, we want to define gate quotient $[u:v]$,

- $[u:u] := 1$,
- For a leaf u & $u \neq v$, $[u:v] := 0$,
- $[u_1+u_2 : v] = [u_1:v] + [u_2:v]$, and

• $[u_1 \times u_2 : v] = [u_1] \times [u_2 : v]$.

▷ $\deg [u:v] \leq \deg u - \deg v$.

▷ If v does <u>not</u> occur in the subcircuit rooted at u, then $[u:v] = 0$.

<u>Proof:</u>

• Inductively, we will reach a leaf u' of u (as no intermediate node is v). At this point as well $u' \neq v$ & so $[u:v] = 0$. □

• <u>Intuition</u> behind $[u:v]$.

Say, $[u] = A \cdot [v] + B$ for some polynomials A, B. We would like to talk about the circuit that computes A.

This is obtained formally by quotienting the $[u]$-subtree by $[v]$. Finally, $[u:v] = A$ (assuming v on the "right" side)

• Which v's should we use?

<u>Defn</u>: The frontier at degree m is
$\mathcal{F}_m := \{ v \mid \deg v_L \leq \deg v_R < m \leq \deg v \}$.

• That is, $\mathcal{F}_m$ are deepest multiplication gates that have $\deg \geq m$.

▷ $u \neq v \in \mathcal{F}_m \Rightarrow [u:v] = 0$.

Pf: • v does not appear in the subcircuit of u. $\square$

• Now, we show how to write a gate in terms of certain quotients.

(Frontier expansion)

<u>Lemma</u>: If $\deg u \geq m$ then $[u] = \sum_{w \in \mathcal{F}_m} [u:w] \times [w]$.

Also, $\deg u \geq m > \deg v \Rightarrow$
$[u:v] = \sum_{w \in \mathcal{F}_m} [u:w] \times [w:v]$.

Proof:

- We do reverse induction on depth(u).
- Base case: u is the deepest, i.e. $u \in \mathcal{F}_m$.

$$\Rightarrow \sum_{w \in \mathcal{F}_m} [u:w]\cdot[w] = [u:u]\cdot[u] + \sum_{u \neq w \in \mathcal{F}_m} [u:w]\cdot[w]$$

$$= 1\cdot[u] + 0 = [u].$$

$$\& \sum_{w \in \mathcal{F}_m} [u:w]\cdot[w:v] = [u:u]\cdot[u:v] + \sum_{u \neq w \in \mathcal{F}_m} [u:w]\cdot[w:v]$$

$$= 1\cdot[u:v] + 0 = [u:v].$$

- Case $u = u_1 + u_2$: $[u] = [u_1] + [u_2]$

$$= \sum_{w \in \mathcal{F}_m} [u_1:w]\cdot[w] + [u_2:w]\cdot[w]$$

$$= \sum_{w \in \mathcal{F}_m} [u_1+u_2:w]\cdot[w]$$

$$\& [u:v] = [u_1:v] + [u_2:v]$$

$$= \sum_{w \in \mathcal{F}_m} [u_1:w]\cdot[w:v] + [u_2:w]\cdot[w:v]$$

$$= \sum_{w \in \mathcal{F}_m} [u_1+u_2:w]\cdot[w:v] \quad .$$