Proof: • The layers are labelled $\ell \in [n+1]$.
Layer $\ell \in [2 \dots n]$ has $\Theta(n^2)$ nodes labelled $v_{i,j}^{(\ell)}$, $i \neq j \in [n]$.
Layer $\ell = 1$ has the node $s = v_{1,1}^{(1)}$ with $wt = x_{1i}$ edge to $v_{1,i}^{(2)}$, $\forall\, 1 < i \leq n$.

• Idea: In $v_{i,j}^{(\ell)}$, $i$ remembers the head & $j$ the current node in the current clow.
With this we intend to hard-code a clow sequence as a path $s \to t$ & vice versa.
For this the ABP has:

1) $\forall i < j \in [n], 1 < \ell < n$, $v_{i,j}^{(\ell)}$ has an edge of $wt = x_{jk}$ to $v_{i,k}^{(\ell+1)}$, for all $k > i$.
[grows the $i$-headed clow from $j$ to $k$.]

2) $\forall i < j \in [n], 1 < \ell < n$, $v_{i,j}^{(\ell)}$ has an edge of $wt = -x_{ji}$ to $v_{k,i}^{(\ell+1)}$, for all $k > i$.

[clow ends, sign changes & new head $= k$.]

3) Last layer: $\forall i<j \in [n]$, $v_{i,j}^{(n)}$ has an edge of $wt = -x_{ji}$ to $t$.

[ clow ends, sign changes & the clow sequence ends. ]

▷ Each ABP path corresponds to a unique clow sequence of $G$. Moreover, the respective weight & signed-weight match.

▷ Each clow sequence of $G$ corresponds to a unique path in ABP.

$\square$

<u>Corollary</u>: 1) $\det_n$ has an $IMM_{O(n^2), O(n)}$.

2) $\det_n$, over any commutative ring, has $O(\lg n)$-depth, unbounded fanin/out, poly$(n)$-size arithmetic circuit.

[det over noncommutative ring?]

# Structural Results

- Arithmetic circuits have some striking "self-reducibilities", that makes studying special cases worthwhile.

Defn: • A polynomial $f$ is homogeneous if all its monomials are equi-degree.
• A circuit is homogeneous if every gate computes a homogeneous polynomial.

Theorem (Homogenization) [Strassen '73]: If $f$ has a circuit $C$ of size $s$. Then, for all $0 \le i < d$, there is a homogeneous circuit $C_i$, of size $O(sd^2)$, that computes the degree $= i$ homogeneous part of $f$.

Proof:

• Wlog, assume $C$ has fanin $\le 2$.
• For any gate $g$, in $C$, we intend to construct gates $g_0, \ldots, g_d$ s.t.

$\forall i \in [0 \dots d-1]$, $g_i$ computes the deg $= i$ homogeneous part of $g$ &
$g_d$ computes the deg $\geq d$ part of $g$.

- We shall construct $g_i$ recursively.
- Let $g$ have children $u$ & $v$.

<u>Case 1</u>: $g = u + v$.

Define $g_i = u_i + v_i$, $\forall 0 \leq i \leq d$.

<u>Case 2</u>: $g = u * v$.

Define $g_i = \sum_{0 \leq j \leq i} u_j * v_{i-j}$, $\forall 0 \leq i < d$

& $g_d = u_0 * (v_d) + u_1 * (v_d + v_{d-1}) + \dots$
$+ u_{d-1} * (v_d + \dots + v_1) + u_d * v$.

- Note that on introducing these extra gates, for each gate $g$ in $C$, we get a circuit $C'$ of size $O(sd^2)$. $\square$