- Thus, det of the adjacency matrix A' of G' is $\det(A') = f$.
- Clearly, A' has dimension $wnd' = O(wdn)$.

$\square$

- More surprising is the converse:

<u>Theorem</u> (Mahajan, Vinay '97): $\det_n$ has a width-$O(n^2)$, depth-$O(n)$ ABP, over any $\mathbb{F}$.

[$\Rightarrow \det_n \in$ VP (depth-$\lg n$) (P-uniform)]
(unbounded fanin/fanout)

- The main tool in the proof is a relaxation of disjoint cycles to <u>closed walks</u> (while still having the det connection).

<u>Defn</u>: Let G be a graph on $V(G) = [n]$.
A <u>clow</u> of G is a closed walk of length, say, ℓ such as $C = (v_1, v_2, .., v_\ell, v_1)$ with v_1 being <u>unique min</u>. <u>head(C)</u> is v_1.

[head does not repeat in a clow.]

A clow sequence is a clow-tuple $(C_1, \dots, C_r)$ with increasings heads, i.e. $\text{head}(C_1) < \dots < \text{head}(C_r)$.

The length of a clow sequence is the sum of the lengths of the underlying clows.

The weight of a clow sequence is the product of the weights of the underlying edges.

The sign of a clow sequence is $(-1)^{\#\text{even-clows}}$. [Even-clow has even length]

▷ A cycle cover is a clow sequence of the same weight & sign.

[Obviously, converse is false.]

- The surprise is:

Lemma [Mahajan, Vinay 1997]: If A is the adjacency matrix of G, then $\det(A) = \sum_{C \in \text{ClowSequence}(G)} \text{sgn}(C) \cdot \text{wt}(C)$.

Proof: • The key idea is to show that the contributions of clow sequences, that are not cycle covers, cancel each other!

• Consider a clow seq. $C = (C_1, \dots, C_r)$ of length l. If C is not a cycle cover then some vertex must repeat.

• Let $i \in [r]$ be the largest such that $C_i = (v_1, v_2, \dots, v_k, v_1)$ has a vertex that repeats (somewhere in $C_i, C_{i+1}, \dots, C_r$).

$\Rightarrow (C_{i+1}, \dots, C_r)$ are disjoint cycles but $(C_i, \dots, C_r)$ are not.

• This can happen in two ways:

Case 1: $\exists\, j' < j \in [k]$, $v_{j'} = v_j$.

Case 2: $\exists\, j \in [k]$, v_j occurs in $C_{i+1}, \dots,$ or C_r.

Over the cases, pick the least j.

• In <u>case 1</u>, vertices $v_{j'+1},\dots,v_j$ are all distinct (as v_j is the first occurrence of a repeated node). So, this gives a <u>cycle</u>.

This cycle is disjoint from $C_{i+1},\dots,C_r$.

• Define a new clow seq. <u>C'</u> by <u>breaking</u> C_i into the clow $(v_1,\dots,v_{j'},v_{j+1},\dots,v_k,v_1)$ & the cycle $(v_{j'+1},\dots,v_j)$.

Note: The heads in C' are distinct.

Note that $wt(C') = wt(C)$ & $sgn(C') = (-1)^{\ell+r+1} = -sgn(C)$.

• In <u>case 2</u>, $v_j \in C_i$ also appears in $C_{i'}$ for $i < i' \le r$. (Note: i' is unique.)

Here we join the clows C_i & $C_{i'}$ at the vertex v_j to get the clow <u>C_i'</u> $:=$

$$(v_1,\dots,v_j, C_{i'}\setminus\{v_j\}, v_{j+1},\dots,v_k,v_1).$$

Note: $v_1 = head(C_i) < head(C_{i'})$

Call the new clow sequence <u>C'</u>.

• Note that $wt(C') = wt(C)$ & $sgn(C') = (-1)^{\ell+r-1} = -sgn(C)$.

▷ The above gives us a map τ from

ClowSequence(G) \ CycleCover(G) to itself such that: τ has no fixed point, it is invertible, & flips the sign.

- Thus, $\displaystyle\sum_{C \in \text{ClowSequence}(G)} \text{sgn}(C) \cdot \text{wt}(C)$ ------(1)

$$= \sum_{C \in \text{CycleCover}(G)} \text{sgn}(C) \cdot \text{wt}(C)$$

$$= \det(A). \qquad \square$$

- The sum over clow sequences can be computed by an ABP:

<u>Lemma</u> [Mahajan, Vinay '97] Expression (1) has a width-n^2, depth-$(n+1)$ ABP, where G has n vertices.

- ABP-width corresponds to memory (<u>registers</u>) & depth corresponds to <u>time</u>.