

ASSIGNMENT 3

POINTS: 50

DATE GIVEN: 26-SEP-2025

DUE: 26-OCT-2025

Rules:

- You are strongly encouraged to work *independently*. That is the best way to understand & master the subject.
- Write the solutions on your own and honorably *acknowledge* the sources if any. <http://cse.iitk.ac.in/pages/AntiCheatingPolicy.html>
- Submit your solutions, before time, to your TA. Please give your LaTeXed or Word processed solution-sheet in PDF. This will be graded, and commented, in-place.
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Problems marked '0 points' are for practice.
- Acknowledgements: Several problems are from *Arora & Barak, Computational Complexity: A Modern Approach*, and other lecture notes.

Question 1:[Chernoff bound for walks] [11 points] Let $G = (V, E)$ be a connected regular undirected graph. Let $f : V \rightarrow \{0, 1\}$ be a boolean function. Let $(v_1, \dots, v_t)$ be a $(t - 1)$ -step *random walk* in G . Prove the following relationship between the mean and the expectation of f .

$$\Pr \left[\frac{1}{t} \sum_{i \in [t]} f(v_i) - \text{Exp}[f] \geq \epsilon + \lambda(G) \right] < e^{-\Omega(\epsilon^2 t)}.$$

Question 2: [8+3 points] Show that for some $\epsilon > 0$, there *exists* a function G that is a $2^{\epsilon n}$ -prg *ignoring* the explicitness condition (of G being E-computable).

Give a complexity upper bound for G as best as you can.

Question 3: [8 points] Show that for every large enough n , there is a boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ whose *average-case-hardness* is exponential.

Question 4: [7 points] Show that if there exists an $S(\ell)$ -prg then there exists a boolean function $f \in E$ such that $H_{\text{wrs}}(f) \geq S(n)$.

Question 5: [13 points] If there exists a boolean function $f \in E$ and $\epsilon > 0$ such that $H_{\text{avg}}(f) \geq 2^{\epsilon n}$, then $\text{MA} = \text{NP}$.

Question 6: [0 points] Show that for d -regular graph G , $\lambda(G) > 1/\sqrt{d}$.

Question 7: [0 points] State and prove the Cauchy-Schwarz inequality. When does the *equality* hold?

Question 8: [0 points] Let A be an $n \times n$ matrix with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. Show that there is a set of orthonormal eigenvectors $\{v_1, \dots, v_n\}$ such that $A = \sum_{i \in [n]} \lambda_i \cdot v_i v_i^T$.

Question 9: [0 points] State and prove the expression for the eigenvalues of A^2 .

Question 10: [0 points] State and prove the expression for the eigenvalues of $A \otimes A$.

Question 11: [0 points] In Reingold's proof done in the class, we converted the given graph G to the expander $G_{O(\log n)}$. Complete the logspace algorithm to find a path from s to t .

Question 12: [0 points] Show that any algorithm of time-complexity $T(n)$ can be converted into a (uniform) circuit family of size $O(T(n)^2)$.

Question 13: [0 points] Why did we prove *two* theorems to upper-bound the spectral-gap of the zig-zag product? Could one follow from the other?

Question 14: [0 points] List the conditions needed in a graph, if you want to explore it by doing a *random walk in Logspace*?

Question 15: *[0 points]* Read about the *Cheeger isoperimetric constant* in complex manifolds. Do you find similarities in the notion of ‘expansion’ in a graph vs that in a surface?

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