CS747 - RANDOMIZED METHODS IN COMPUTATIONAL COMPLEXITY
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ASSIGNMENT 2

POINTS: 50

DATE GIVEN: 05-SEP-2025 DUE: 25-SEP-2025

Rules:

- You are strongly encouraged to work *independently*. That is the best way to understand & master the subject.
- Write the solutions on your own and honorably *acknowledge* the sources if any. `http://cse.iitk.ac.in/pages/AntiCheatingPolicy.html`
- Submit your solutions, before time, to your TA. Please give your LaTeXed or Word processed solution-sheet in PDF. This will be graded, and commented, in-place.
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Problems marked '0 points' are for practice.
- Acknowledgements: Several problems are from *Arora & Barak, Computational Complexity: A Modern Approach*, and other lecture notes.

Question 1: *[6+3 points]* In the class we proved the Razborov-Smolensky result: $\text{mod}_p \notin \text{ACC}^0[q]$, for $p = 2 = q - 1$. Give the general proof for primes $p \neq q$.

What happens to the proof when one of them is a composite?

Question 2: *[7 points]* Show that the OR of n variables cannot be expressed as a polynomial over $\mathbb{F}_p$ of degree less than n.

Question 3: *[13 points]* Give an example of a set-family $\mathcal{Z}$ with $(p-1)^\ell$ subsets (each size $\leq \ell$), such that there is no sunflower with p-petals.

Question 4: *[6+9 points]* Let A be a real symmetric stochastic matrix. Show that

(1) its eigenvalues are real with absolute value at most 1.
(2) its eigenvectors, for different eigenvalues, are linearly independent.

Question 5: *[6 points]* Let A, B be real symmetric stochastic matrices. Show that the *second-largest eigenvalue* function $\lambda(\cdot)$ is *subadditive* (i.e. $\lambda(A+B) \leq \lambda(A) + \lambda(B)$).

Question 6: *[0 points]* Let G be a graph with only *two* distinct eigenvalues. Prove that G is a *complete graph*.

Question 7: *[0 points]* Show that for an $n \times n$ matrix $X = ((x_{i,j}))$, the polynomial $\text{per}(X)$ has a formula of size $2^{O(n)}$.

Question 8: *[0 points]* Show that $\text{AM}[k] = \text{AM}[2]$ for all constant $k \geq 2$.

Question 9: *[0 points]* What big change happens in the above proof, about $\text{AM}[k]$, if k is *non*-constant?

Question 10: *[0 points]* Show that the graph isomorphism problem is in NP∩coAM. (So, "almost" in NP∩coNP.)

Is there a better (algorithmic) result known for graph isomorphism?

Question 11: *[0 points]* Make a list of interesting problems that are in AM∩coAM. (So, "almost" in NP∩coNP.)

Could any such problem be NP-*hard* ?

Question 12: *[0 points]* Fix parameters n, d. Show that a random n-vertex d-regular graph is a $(n, 2d, 1/3)$-combinatorial expander.

Question 13: *[0 points]* Consider a subset A of the n-tuples $\mathbb{F}_3^n$. We are interested in the largest set A that is *free of arithmetic progressions of size* 3 (i.e. there is no $a, b, c \in A$ satisfying $a + b + c \equiv \mathbf{0} \mod 3$).

Could you relate this question to the *sunflower lemma*?

Question 14: *[0 points]* Recall our proof of the $n^{\Omega(\sqrt{k})}$ monotone circuit lower bound for the k-clique problem. Is this bound tight?

Question 15: *[0 points]* Think of another "hard" problem that is monotone.

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