Zero Knowledge Proof through Graph 3-Coloring

DIVYANSH AGARWAL

CS747: Randomized Methods in Computational Complexity

Indian Institute of Technology, Kanpur

Introduction to Zero Knowledge

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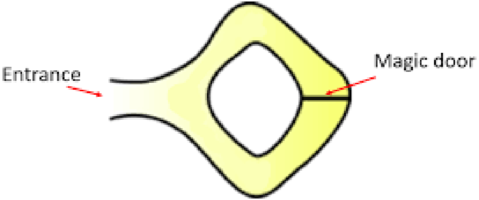

Interactive Proof System

A pair of interactive machines (P, V) is called an **interactive proof system** for a language L if V is PPT TM and the following two conditions hold:

- **Completeness:** For every $x \in L$, $\exists\ P$

$$\Pr[\langle P, V\rangle(x) = 1] \geq \frac{2}{3}$$

- **Soundness:** For every $x \notin L$ and every interactive machine B,

$$\Pr[\langle B, V\rangle(x) = 1] \leq \frac{1}{3}$$

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1. With probability at most $\frac{1}{2}$, on input x, machine M^* outputs a special symbol $\perp$, i.e.,

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where, $m^*(x)$ denote the random variable describing the distribution of $M^*(x)$ conditioned on $M^*(x) \neq \perp$, i.e.,

$$\Pr[m^*(x) = \alpha] = \Pr[M^*(x) = \alpha \mid M^*(x) \neq \perp], \quad \forall \alpha \in \{0, 1\}^*.$$

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$\text{view}_{V^*}^{P}(x) :=$ the random variable describing the content of the random tape of V^* and the messages V^* receives from P during their interaction on common input x.

We say that (P, V) is **zero-knowledge** if for every PPT TM V^*, there exists PPT algorithm M^* such that the ensembles

$$\{\text{view}_{V^*}^{P}(x)\}_{x \in L} \quad \text{and} \quad \{M^*(x)\}_{x \in L}$$

are **computationally indistinguishable.**

Computational Indistinguishable and Complexity Class Relation

We consider ensembles indexed by strings from a language L. We say that the ensembles $\{R_x\}_{x \in L}$ and $\{S_x\}_{x \in L}$ are **computationally indistinguishable** if for every probabilistic polynomial-time algorithm D, for every polynomial $p(\cdot)$, and for all sufficiently long $x \in L$, it holds that

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Complexity Class Relations:

$$\text{BPP} \subseteq \text{PZK} \subseteq \text{CZK} \subseteq \text{IP}$$

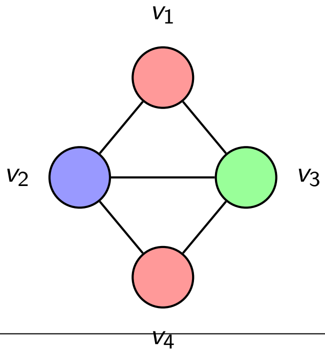
Graph 3-Coloring (G3C)

The language **Graph 3-Coloring**, denoted $G3C$, consists of all simple (finite) undirected graphs that can be vertex-colored using three colors such that no two adjacent vertices are given the same color.

Formally, a graph $G = (V, E)$ is **3-colorable** if there exists a mapping

$$\phi : V \to \{1, 2, 3\}$$

such that $\phi(u) \neq \phi(v)$ for every $(u, v) \in E$.



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- Graphs that are not 3-colorable must contain at least one "bad" edge (same color), so they are rejected with noticeable probability.
- The zero-knowledge property holds intuitively because a simulator can mimic the interaction by picking random distinct colors for the verifier's chosen edge.

Properties:

- If G is 3-colorable, verifier always accepts.
- If not, verifier rejects with probability at least $1/|E|$.
- Repeating the protocol increases confidence.

However, for implementation, we don't have perfect boxes, hence we use commitment schemes.

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Phase 1 — Commit (Hiding Phase)

- The sender (prover) chooses a secret value v and some random coins r.
- It computes a **commitment** $C = \text{Commit}(v; r)$ and sends C to the receiver (verifier).
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$$\textbf{Prover} \quad \begin{array}{c} \xrightarrow{\quad\text{Commit } C\quad} \\ \xrightarrow{\quad\text{Reveal } (v, r)\quad} \end{array} \quad \textbf{Verifier}$$

Commitment Scheme

A **commitment scheme** is a two-phase protocol between:

- **Sender (S):** commits to a value $v \in \{0, 1\}$ using randomness r.
- **Receiver (R):** obtains commitment C and later verifies v.

Requirements:

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- **Binding:** For almost all random coins r of R, there exists at most one message m forming a valid opening.

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- Uses the **commitment scheme** to commit to each color:

$$s_1, \dots, s_n \leftarrow \{0,1\}^n, \quad c_i = C_{s_i}(\phi(i))$$

- Sends $c_1, \dots, c_n$ to the verifier.

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P2: The prover reveals colors of u and v by sending $(s_u, \phi(u))$, $(s_v, \phi(v))$.

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V2: The verifier checks:

$$c_u = C_{s_u}(\phi(u)), \quad c_v = C_{s_v}(\phi(v)), \quad \phi(u) \neq \phi(v)$$

Accept if all hold; otherwise reject.

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The next slides will prove computational zero knowledge for above protocol.

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4. **Reveal simulated colors:**
   - If $e_u \neq e_v$, output simulated transcript $(G, r, (c'_1, \ldots, c'_n), (s_u, e_u, s_v, e_v))$.
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- **Claim 1:** Using request-obliviousness, simulator's failure probability $\approx \frac{1}{3}$.
- **Claim 2:** Conditioned on success, simulator's output is computationally indistinguishable from the real verifier's view (else we break **"nonuniform"** secrecy).

Conclusions

- The simulator M^* succeeds with constant probability ($\approx 2/3$) even without knowing the 3-coloring, by exploiting the hiding property of the commitment scheme.
- Hence, We achieve **computational zero-knowledge** for 3-Colorability: the NP complete problem.

References

- Oded Goldreich, *Foundations of Cryptography, Volume 1: Basic Tools*, Cambridge University Press, 2001.
- Oded Goldreich, Silvio Micali, and Avi Wigderson, *Proofs that Yield Nothing But Their Validity, or All Languages in NP Have Zero-Knowledge Proof Systems*, Journal of the ACM, 1986.