CS747 - RANDOMIZED METHODS IN COMPUTATIONAL COMPLEXITY
NITIN SAXENA

ASSIGNMENT 3

POINTS: 50

DATE GIVEN: 04-OCT-2023 DUE: 25-OCT-2023

Rules:

- You are strongly encouraged to work *independently*. That is the best way to understand & master the subject.
- Write the solutions on your own and honorably *acknowledge* the sources if any. `http://cse.iitk.ac.in/pages/AntiCheatingPolicy.html`
- Submit your solutions, before time, to your TA. Please give your LaTeXed or Word processed solution-sheet in PDF. This will be graded, and commented, in-place.
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Problems marked '0 points' are for practice.
- Acknowledgements: Several problems are from *Arora & Barak, Computational Complexity: A Modern Approach*, and other lecture notes.

Question 1:[Chernoff bound for walks] *[11 points]* Let $G = (V, E)$ be a connected regular undirected graph. Let $f : V \to \{0, 1\}$ be a boolean function. Let $(v_1, \ldots, v_t)$ be a $(t-1)$-step *random walk* in G. Prove the following relationship between the mean and the expectation of f.

$$\Pr\left[\frac{1}{t}\sum_{i\in[t]} f(v_i) - \mathrm{Exp}[f] \;\geq\; \epsilon + \lambda(G)\right] \;<\; e^{-\Omega(\epsilon^2 t)}\,.$$

Question 2: *[8+3 points]* Show that for some $\epsilon > 0$, there *exists* a function G that is a $2^{\epsilon n}$-prg *ignoring* the explicitness condition (of G being E-computable).

Give a complexity upper bound for G as best as you can.

Question 3: *[8 points]* Show that for every large enough n, there is a boolean function $f : \{0,1\}^n \to \{0,1\}$ whose *average*-case-hardness is exponential.

Question 4: *[7 points]* Show that if there exists an $S(\ell)$-prg then there exists a boolean function $f \in \mathrm{E}$ such that $\mathrm{H}_{\mathrm{wrs}}(f) \geq S(n)$.

Question 5: *[13 points]* If there exists a boolean function $f \in \mathrm{E}$ and $\epsilon > 0$ such that $\mathrm{H}_{\mathrm{avg}}(f) \geq 2^{\epsilon n}$, then MA = NP.

Question 6: *[0 points]* Show that for d-regular graph G, $\lambda(G) > 1/\sqrt{d}$.

Question 7: *[0 points]* State and prove the Cauchy-Schwarz inequality. When does the *equality* hold?

Question 8: *[0 points]* Let A be an $n \times n$ matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Show that there is a set of orthonormal eigenvectors $\{v_1, \ldots, v_n\}$ such that $A = \sum_{i\in[n]} \lambda_i v_i v_i^{\mathrm{T}}$.

Question 9: *[0 points]* State and prove the expression for the eigenvalues of A^2.

Question 10: *[0 points]* State and prove the expression for the eigenvalues of $A \otimes A$.

Question 11: *[0 points]* In Reingold's proof done in the class, we converted the given graph G to the expander $G_{O(\log n)}$. Complete the logspace algorithm to find a path from s to t.

Question 12: *[0 points]* Show that any algorithm of time-complexity $T(n)$ can be converted into a (uniform) circuit family of size $O(T(n)^2)$.

Question 13: *[0 points]* Why did we prove *two* theorems to upper-bound the spectral-gap of the zig-zag product? Could one follow from the other?

□□□