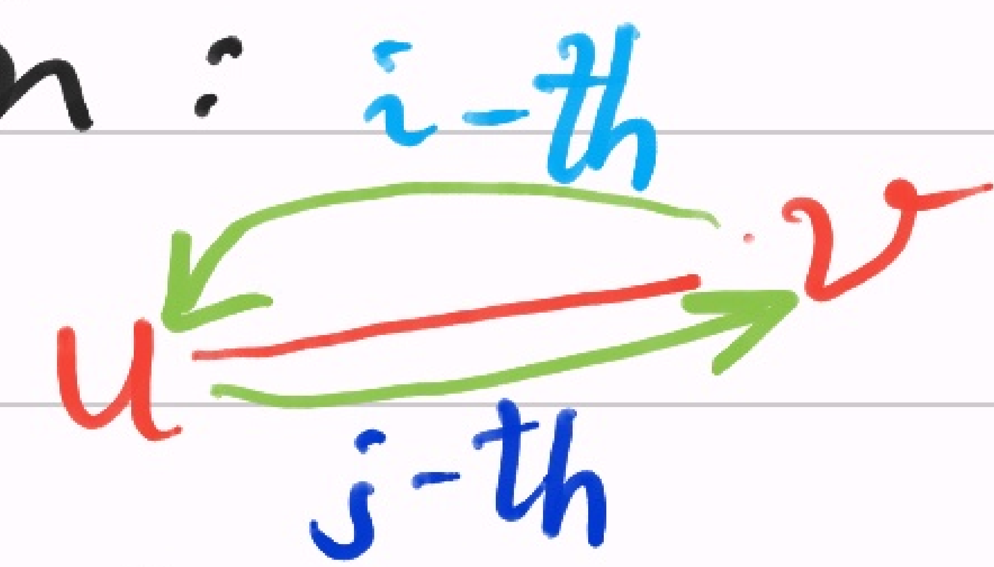
# Explicit Expander Constructions

- We will construct an explicit expander family.
- Idea is to use 4 kinds of graph products.

  Tradeoff: between degree & spectral-gap.
  (we want constants)

- We'll need a new graph representation:




**Defn:** If $G$ is an $n$-vertex $d$-degree connected graph then we label each neighbor of each vertex using $[d]$. Define a rotation map

$\hat{G} : [n] \times [d] \to [n] \times [d]$ ; $(v, i) \mapsto (u, j)$

where $u$ is the $i$-th vertex of $v$ & $v$ $j$-th vertex of $u$.

▷ $\hat{G}$ is a permutation of $[n]\times[d]$.

Pf: $\because$ $\hat{G}$ is 1-1 & onto. $\square$

- Given $G$ & $G'$ (in the above representation) we'll define 4 products:
  1. Path product
  2. Tensor product
  3. Replacement product
  4. Zig-Zag product

▷ Combining them we'll improve expansion!

## Matrix Product (or Path product)

Defn: For two n-vertex graphs $G, G'$ with degrees $d, d'$, the graph $GG'$ is the one with the normalized adjacency matrix $A \cdot A'$.

Proposition: 1) $AA'$ is stochastic.

2) $GG'$ may have repeated edges, $\deg = dd'$, #vertices $= n$.

($\mathbb{1}$ = all-one)

Pf: 1) $AA' \cdot \mathbb{1} = A \cdot \mathbb{1} = \mathbb{1} \Rightarrow AA'$ row-sum is 1.

$\Rightarrow$ (by symmetry) $AA'$ is stochastic.

2) $\forall u \in V(G)$, $u$ has $d$ neighbors $v$. $\forall v \in V(G')$ has $d'$ neighbors.

$\Rightarrow$ Neighbors of $u \in V(GG')$ are $d \times d'$. $\square$

**Claim:** $\lambda(GG') \leq \lambda(G) \cdot \lambda(G')$. $\leftarrow$ spectral gap increases

**Proof:** • $\lambda(GG') = \max_{u \in \mathbb{1}^\perp} \frac{\|AA'u\|}{\|u\|} = \max \frac{\|AA'u\|}{\|A'u\|} \cdot \frac{\|A'u\|}{\|u\|}$

spectral norm

$[\langle A'u, \mathbb{1} \rangle = \langle u, A'\mathbb{1} \rangle = \langle u, \mathbb{1} \rangle = 0]$ $\leq \lambda(A) \cdot \lambda(A')$. $\square$

**Theorem:** $G, G'$ are $(n, d, \lambda)$, $(n, d', \lambda')$-expanders

$\Rightarrow$ $GG'$ is $(n, dd', \lambda\lambda')$-expander.

– Thus, matrix product improves the spectral gap at the cost of the degree.

# Tensor Product

**Defn:** The graph $\underset{n\text{-vertex}}{G} \otimes \underset{n'\text{-vertex}}{G'}$ is the one with the normalized adjacency matrix $A \otimes A'$.

where, $A \otimes A' := \begin{pmatrix} A_{11}A' & \cdots & A_{1n}A' \\ \vdots & & \vdots \\ A_{n1}A' & \cdots & A_{nn}A' \end{pmatrix}_{nn' \times nn'}$.

**Proposition:** 1) $A \otimes A'$ is symmetric stochastic.

2) $G \otimes G'$ is $nn'$-vertex, $dd'$-degree.

**Pf:** 1) $(A \otimes A') \cdot 1_{nn'} = (A \otimes A') \cdot (1_n \otimes 1_{n'}) = A \cdot 1_n \otimes A' 1_{n'}$

$$= 1_n \otimes 1_{n'} = 1_{nn'}.$$

2) $u$-th "row" is $(A_{u1}A', \cdots, A_{un} \cdot A')$. Each row has wt. $d \times d'$. $\square$

Claim: $\lambda(G \otimes G') = \max(\lambda(G), \lambda(G'))$.

Pf: • Let $1 = \lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ &

$1 = \lambda_1' \geq |\lambda_2'| \geq \cdots \geq |\lambda_n'|$ be eigenvalues of $A$ & $A'$ resp.

• $[(A \otimes A') \cdot (v \otimes v') = (Av) \otimes (A'v')] \Rightarrow$ eigenvalues of $A \otimes A'$ are: $\{\lambda_i \lambda_j' \mid (i,j) \in [n] \times [n']\}$.

• Thus, the largest ones, apart from 1, are in: $\{\lambda_i \mid i \in [2..n]\} \cup \{\lambda_j' \mid j \in [2..n]\}$.

$\Rightarrow \lambda(G \otimes G') = \max(\lambda(G), \lambda(G'))$. $\square$

Theorem: $G, G'$ are $(n,d,\lambda), (n',d',\lambda')$-expanders resp.
$\Rightarrow G \otimes G'$ is $(nn', dd', \max(\lambda,\lambda'))$-expander.

- Tensor product increases #vertices (& degree), while preserving the spectral-gap.

## The Replacement Product

- We want to reduce the degree.
- This product is easier seen as a walk rather than a matrix operation.

Idea: Use an expander $G'$ with $D$-vertex & $d$-degree, to pick the neighbor in a $D$-degree graph $G$. $(d \ll D)$

- This is similar to the idea to reduce random-bits.
  - This motivates the product $G \circledR G'$:

**Defn:** Let $G, G'$ be graphs with vertices $n, D$ & degrees $D, d$ & normalized adj. matrices $A, A'$ resp.
$H := G \circledR G'$ is $(nD)$-vertex graph s.t.

i) $\forall u \in V(G)$, $H$ has a copy of $G'$, say $H_u$, called a cloud. I.e. $\forall i \in V(G')$, $(u,i) \in V(H)$; and is called the $i$-th vertex in $u$-th cloud.

ii) For $(i,j) \in E(G')$, put $\forall u \in V(G), \big((u,i),(u,j)\big) \in E(H)$.

iii) If $\hat{G}(u,i) = (v,j)$ then put $((u,i),(v,j)) \in E(H)$.

**Claim:** $H$ is $nD$-vertex, $(d+1)$-degree.

Pf: • $V(H) = V(G) \times V(G')$.

- Neighbors of $(u,i) \in V(H)$ are $d$-many in the cloud $H_u$. Plus, one more given by $\hat{G}(u,i)$.

$\Rightarrow \deg(H) = d+1$. $\square$

– Replacement product reduces degree $D$ to $d+1 \ll D$.

## Zig-Zag Product

– Idea: Consider length-3 paths in $G ® G'$ that zig-zag the clouds. ( cloud → out → another cloud)

Defn: On the vertex-set $V(G ® G')$ define zig-zag $H := G ⓩ G'$ s.t. $((u,i),(v,j)) \in E(H)$ if $\exists \ell, k$ with $((u,i),(u,\ell)), ((u,\ell),(v,k)), ((v,k),(v,j)) \in E(G ® G')$.

▷ $G \circledz G'$ is $nD$-vertex, $d^2$-degree. $[d^2 \ll D]$

Pf: • #neighbors of $(u,i)$ in $G \circledz G' = d \times 1 \times d = d^2$. $\square$

▷ Its normalized adj. matrix is $A \circledz A' := B\hat{A}B$,

where $\hat{A}[(u,i),(v,j)] := \begin{cases} 1, & \text{if } \hat{G}(u,i) = (v,j) \\ 0, & \text{else} \end{cases}$

& $B[(u,i),(v,j)] := \begin{cases} A'[i,j], & u = v \\ 0, & u \neq v \end{cases}$

Pf: • $A \circledz A'$ encodes the defn. of $G \circledz G'$. $\square$

• Note: $(A \circledz A') \cdot 1 = B\hat{A}B \cdot 1 = B\hat{A} \cdot 1 = B \cdot 1 = 1$.

▷ $A \circledz A'$ is stochastic symmetric.

# Spectral analysis of Zig-Zag

**Theorem** (Reingold, Vadhan, Wigderson '02): $\lambda(G)=a$ & $\lambda(G')=b \implies \lambda(G \circledz G') \le a+2b+b^2$.

**Proof**: • Let $M := A \circledz A'$.

• Recall: $M = B\hat{A}B$ ; $\hat{A}$ is permutation & $B = I_n \otimes A'$.

• Write $B =: I_n \otimes J/D + I_n \otimes \underbrace{(A' - J/D)}_{E}$, where $J$ is the all-one matrix.

• Define $\bar{J} := I_n \otimes J/D$ & $\bar{E} := I_n \otimes E$.

$\Rightarrow M = B\hat{A}B = (\bar{J}+\bar{E})\cdot\hat{A}\cdot(\bar{J}+\bar{E}) = \bar{J}\hat{A}\bar{J} + \bar{J}\hat{A}\bar{E} + \bar{E}\hat{A}\bar{J} + \bar{E}\hat{A}\bar{E}$.

• Each of these four we'll upper-bound the matrix norm: $\lambda(A) := \|A\| := \max_{x \in \bar{1}^\perp} \|Ax\| / \|x\|$. ~ 2nd largest eigenval.

• Let $\underline{\mathcal{L}(A)} := \max_x \|Ax\| / \|x\|$ ← largest eigenval

▷ $\|E\| \leq \mathcal{L}(E) = \mathcal{L}(A' - J/D)$.

• Let $\lambda_1, \ldots, \lambda_D$ be the eigenvalues of $A'$ s.t.

$$1 = \lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_D|.$$

▷ Write $A' = \sum_{i \in [D]} \lambda_i \cdot v_i v_i^T$, for orthonormal eigenvectors $\{v_i\}_i$ of $A'$.

$$\Rightarrow \|E\| \leq \mathcal{L}(A' - J/D) = \mathcal{L}(A' - \lambda_1 \cdot v_1 v_1^T) = \mathcal{L}\left(\sum_{i>1}^{D} \lambda_i \cdot v_i v_i^T\right)$$

$\left[ \|\sum_{i>1} \lambda_i \cdot v_i v_i^T \cdot \bar{x}\|^2 = \|\sum_{i>1} \lambda_i \cdot v_i \cdot \langle v_i, \bar{x} \rangle\|^2 = \sum \|\lambda_i \cdot v_i \cdot \langle v_i, \bar{x} \rangle\|^2 \right.$

Say, $\bar{x} = \sum \alpha_i v_i \Rightarrow \text{RHS} = \sum_{i>1} \lambda_i^2 \cdot \alpha_i^2 \leq \lambda_2^2 .\ \Big]$

$$\leq |\lambda_2| = \|A'\| = b.$$

• Going back, we've by triangle-inequality:

$$\|M\| \leq \|\bar{J}\hat{A}\bar{J}\| + \|\bar{J}\hat{A}E\| + \|E\hat{A}\bar{J}\| + \|E\hat{A}E\|$$

$$\leq \quad " \quad + \|\bar{J}\|\cdot\|\hat{A}\|\cdot\|E\| + \|E\|\cdot\|\hat{A}\|\cdot\|\bar{J}\| + \|E\|\cdot\|\hat{A}\|\cdot\|E\|.$$

[$\because$ $\hat{A}$, $\bar{J}$, $E$ map $1^\perp$ to itself.]

$$\leq \|\bar{J}\hat{A}\bar{J}\| + 1\cdot 1\cdot b + b\cdot 1\cdot 1 + b\cdot 1\cdot b$$

$$= \quad " \quad + (2b + b^2).$$

• Now consider $\bar{J}\hat{A}\bar{J}$:

▷ $\bar{J}\hat{A}\bar{J} = A \otimes J/D$.

Pf: • $(\bar{J}\hat{A}\bar{J})_{(u,i),(v,j)} = (\bar{J})_{(u,i),-} \cdot \hat{A} \cdot (\bar{J})_{-,(v,j)} = \dfrac{A_{u,v}}{D}$ (Why?)

$$= (A \otimes J/D)_{(u,i),(v,j)}. \qquad \square$$

- Note that the eigenvalues of $J$ are $\{1, 0, 0, \ldots, 0\}$. (Why?)

[$\because$ $J \cdot \mathbb{1} = \mathbb{1}$ & $\forall \bar{x} \in \mathbb{1}^{\perp}$, $J \cdot \bar{x} = 0$.]

$\Rightarrow \|A \otimes J/D\| = \lambda(A) = \|A\| = a$.

$\Rightarrow \|\tilde{J} \hat{A} \tilde{J}\| = a$.

$\Rightarrow \|M\| \le (a + 2b + b^2)$. $\square$

**Theorem:** $G, G'$ are $(n, D, \lambda_1)$, $(D, d, \lambda_2)$-expanders

$\Rightarrow G \textcircled{z} G'$ is $(nD, d^2, \lambda_1 + 2\lambda_2 + \lambda_2^2)$-expander.

- Thus, '$\textcircled{z}$' reduces the degree dramatically (if $d \ll D$), without increasing the $\lambda$ too much!

## The Construction

- Using the products we give a strongly explicit family of expanders,
  i.e. given $(u,i)$, the $i$-th neighbor of $u$, is $\mathrm{poly}(\lg|V|)$-time computable.

**Theorem:** $\exists$ strongly explicit $(d^2,\lambda)$-expander-family for $\infty$-ly many constants $d\in\mathbb{N}$ & $\lambda\in(0,1)$.

Pf: • We'll recursively construct $\{G_k\}_{k\geq 1}$ s.t. $G_k$ has $2^{O(k)}$ vertices.

• Let $H$ be a $(d^8, d, 0.04)$-expander.

[H could be found randomly, or by known constructions.]

- Let $G_1 := H^2$ be $(d^8, d^2, 0.04^2)$-expander.
- Let $G_2 := G_1$.
- For odd $k \geq 3$, $G_{k+1} := G_k := \left(G_{\frac{k-1}{2}} \otimes G_{\frac{k-1}{2}}\right)^2$ ⓩ $H$.

Claim: $\forall$ odd $k$, $G_k$ is $(d^{8k}, d^2, 0.1)$-expander.

Pf: • True for $k=1$.

- By induction, $\#V(G_k) = d^{8\cdot\frac{k-1}{2}\times 2}\cdot d^8 = d^{8k}$.
- $\deg\left(G_{\frac{k-1}{2}}^{\otimes 2}\right)^2 = (d^{2\times 2})^2 = d^8$.

$\Rightarrow \deg($ ___ ⓩ $H) = d^2$.

- Finally, $\lambda(G_k) \leq (0.1)^2 + 2\times(0.04) + (0.04)^2 < 0.1$. $\square$

- The algorithm to find neighbors, in $G_k$, of $v \in V(G_k)$ is also recursive:

  Say, listing a row in $G_{\frac{k-1}{2}}$ takes $T\left(\frac{k-1}{2}\right)$ time.

$\Rightarrow$ Listing a row in $G_{\frac{k-1}{2}}^{\otimes 2}$ takes $2 \cdot T\left(\frac{k-1}{2}\right)$.

$d^4$-sparse

$\Rightarrow$ Listing all the needed $d^4$-rows in $G_{\frac{k-1}{2}}^{\otimes 2}$ (in zig-zag product) takes $d^4 \cdot 2 \cdot T\left(\frac{k-1}{2}\right)$ time

$\Rightarrow$ Listing a row in $G_k$ takes time $T(k) = O\left(d^4 \cdot T\left(\frac{k-1}{2}\right)\right)$.

$$\Rightarrow \quad T(k) = O(d^{4 \lg k}) = O(k^{4 \lg d}) = \mathrm{poly}(k)$$

$$= \mathrm{poly.log}(\#V(G_k))$$

[$\because$ $d$ is constant]. $\square$

- To derandomize "Upath $\in$ RL", we'll need operations to make input $G$ an expander, with changing the connectivity!

- To use zig-zag product we'll need another estimate on $\lambda(G \text{ ⓩ } G')$.

- For the input graph, $\lambda(G)$ may be large in general ($\approx 1 - 1/\text{poly}(n)$).
  To make it small, we prove a different bound.

<u>Theorem</u> (Reingold, Trevisan, Vadhan '05): $\lambda(G) = 1-\varepsilon$ & $\lambda(G') = 1-\delta \implies \lambda(G \textcircled{z} G') \le 1-\varepsilon\delta^2$. ← multiplicative

<u>Proof</u>: • As before, $\underline{M} := A \textcircled{z} A' = B\hat{A}B$, where $\underline{B} = I_n \otimes A'$ & $\underline{\hat{A}}$ is rotation map of $G$.

• Express $A'$ as $\delta \cdot J/D + (1-\delta)\cdot C$, where $J$ = all-one & $C$ is a matrix s.t. $\|C\| \le 1$.

[Pf: • $A' = \sum_{i\in[D]} \lambda_i \cdot v_i v_i^T \implies \|(A' - \delta\cdot J/D)\cdot \bar{x}\|$ ← $\bar{x} \in \mathbb{1}^\perp$

$= \|((1-\delta)\cdot J/D + (1-\delta)\cdot v_2 v_2^T + \cdots)\cdot \bar{x}\|$

$\le \|(1-\delta)\cdot v_2 \cdot \langle v_2, \bar{x}\rangle + \lambda_3 \cdot v_3 \cdot \langle v_3, \bar{x}\rangle + \cdots\|$

$\le (1-\delta)\cdot \|\bar{x}\|$ [∵ $v_i$'s form <u>orthonormal basis</u>]

$\implies \left\|\frac{1}{(1-\delta)}\cdot(A' - \delta\cdot J/D)\right\| \le 1.$ □ ]

$$\Rightarrow \quad B = I_n \otimes \left(\delta \cdot J/D + (1-\delta) \cdot C\right) =: \delta \cdot \overline{J} + (1-\delta) \cdot \overline{C}$$

$$\Rightarrow \quad M = B \hat{A} B = \left(\delta \overline{J} + (1-\delta)\overline{C}\right) \cdot \hat{A} \cdot \left(\delta \overline{J} + (1-\delta)\overline{C}\right)$$

$$=: \delta^2 \cdot \overline{J} \hat{A} \overline{J} + (1-\delta^2) \cdot F$$

where $F := \frac{\delta}{1+\delta} \cdot \left(\overline{J}\hat{A}\overline{C} + \overline{C}\hat{A}\overline{J}\right) + \frac{1-\delta}{1+\delta} \cdot \overline{C}\hat{A}\overline{C}$

$\overline{J} := I_n \otimes J/D$ & $\overline{C} := I_n \otimes C$.

$\triangleright$ $\|F\| \leq 1$. [Pf: $\|\overline{C}\| \leq 1$, $\|\overline{J}\| \leq 1$ & $\hat{A}$ is permutation.]

$$\Rightarrow \quad M = \delta^2 \cdot (A \otimes J/D) + (1-\delta^2) \cdot F$$

$$\Rightarrow \quad \lambda(M) = \|M\| \leq \delta^2 \cdot \|A \otimes J/D\| + (1-\delta^2) \cdot \|F\|$$

$$\leq \underbrace{\delta^2 \cdot (1-\varepsilon)}_{\text{-ve part}} + \underbrace{(1-\delta^2) \cdot 1}_{\text{close to 1!}} = 1 - \varepsilon\delta^2. \qquad \square$$

**Theorem (Reingold '05):** Upath $\in \mathbb{L}$.

**Proof:** • Let $G$ be the given graph; undirected, $n$-vertex ($s$ = start-vertex).

Idea: Apply the graph-products on $G$ to get $\tilde{G}$ s.t. the connected component of $s$, in $\tilde{G}$, is an expander (with $\lambda, d$ = constant).

$\Rightarrow$ Shortest-paths in this connected-component of $\tilde{G}$ have length $\leq O(\lg n)$.

$\Rightarrow$ Guessing in logspace possible!

• Let $d$ be a large constant s.t. $(d^{16}, d, 0.5)$-expander $H$ exists.

- Wlog assume $G$ to be regular of $\deg = D := d^{16}$. (& connected)
- Now, transform $G$ as follows:

$$G_0 := G, \quad G_{i+1} := (G_i \circledZ H)^8, \text{ for } i \geq 0.$$

compute only locally, in logspace.

$\triangleright$ $G_k$ is $(nd^{16k}, \overset{=:D}{d^{16}})$-graph.

Pf: • By induction, $\#V(G_k) = nd^{16(k-1)} \cdot d^{16} = nd^{16k}$.

• $\deg(G_k) = (d^2)^8 = d^{16}$. $\square$

- Recall that for any connected graph $G$,

$$\lambda(G) \leq 1 - \frac{1}{8Dn^3} = 1 - \left(\frac{1}{8d^{16} \cdot n^3}\right).$$

$\triangleright$ $\lambda(G_{k-1}) =: 1-\varepsilon \Rightarrow \lambda(G_k) \leq \left(1-\varepsilon\cdot\frac{1}{4}\right)^8$

$$\Rightarrow 1-\lambda(G_k) \geq 1-(1-\varepsilon/4)^8 = 8\cdot\left(\frac{\varepsilon}{4}\right) - \frac{8\cdot 7}{2}\cdot\left(\frac{\varepsilon}{4}\right)^2 + \cdots$$
$$= 2\varepsilon - \frac{7}{4}\varepsilon^2 + \cdots = \varepsilon\cdot\left(2-\frac{7\varepsilon}{8}\right) + \cdots$$

[ Note: $\varepsilon < 1/2 \Rightarrow \cdots > \varepsilon\cdot\left(2-\frac{7}{16}\right) = \varepsilon\cdot\frac{25}{16} > \varepsilon$. ]

$\triangleright$ $1-\lambda(G_k) > (1-\lambda(G_{k-1}))\cdot(\text{constant}>1)$.

$\Rightarrow$ For $\ell := O(\lg n)$, $1-\lambda(G_\ell) \geq 1/2$.

$\Rightarrow \triangleright$ $G_\ell$ has $O(\lg n)$-length shortest-paths & $\deg = d^{16}$.

Exercise: Do the walk in $G_\ell$ in logspace.

[ Use Reingold's recursive data structure. ]