CS688 - COMPUTATIONAL ARITHMETIC-GEOMETRY & APPLICATIONS

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MID-SEMESTER EXAMINATION

POINTS: 75

DATED: 22-FEB-2023 DUE: 24-FEB-2023 (10PM)

Rules:

- You are strongly encouraged to work *independently*. You are not allowed to discuss.
- Write the solutions on your own and honorably *acknowledge* the sources if any. cse.iitk.ac.in/pages/AntiCheatingPolicy.html
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Submit your solutions, before time, to your TA. Preferably, submit a printed/pdf copy of your LaTeXed or Word processed solution sheet.
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Question 1: [*6+9 points*] Recall the notion of morphisms on the affine or projective line.
- Show that *two points fix an automorphism* on the affine line, i.e. for any distinct $P_1, P_2 \in \mathbb{A}^1_k$ there exists a *unique* automorphism φ of $\mathbb{A}^1_k$ such that $\varphi(P_i) = P_i$, $i \in [2]$.
- How many points fix an automorphism on the projective line $\mathbb{P}^1_k$? Give a proof.

Question 2: [*10 points*] Cluster the following (quasi-)affine varieties, in $\mathbb{A}^2_k$, up to *birational* equivalence. Give enough justifications.

(1) $Z(y^2 - x^3 + x^2)$
(2) $Z(x^3 - x) \setminus Z(x^2 - 1)$
(3) $Z(y - x^3 + x)$
(4) $Z(y^2 - x^3 + x)$
(5) $Z(y^2 - x + 1)$

Question 3: [*18+7 points*] Consider $A = k[x_1, \ldots, x_n]$ for any field k. Let $f, f_1, \ldots, f_m \in A$ be up to degree d. We are interested in an *effective* version of the containment $f \in \langle f_1, \ldots, f_m \rangle =: \mathcal{I}$.

- Prove that: If $f \in \mathcal{I}$ then $f =: \sum_{i=1}^{m} a_i f_i$, for $a_i \in A$ having degree $(nmd)^{2^{O(n)}}$.

[*Hint*: Induction on n and easy linear algebra.]

- Using this degree bound we get a "brute-force" algorithm for the ideal membership problem. What is its complexity (time/space)?

Question 4: [*5*5 points*] Here, we will "correct" the definition of the affine space $\mathbb{A}_k^n$ seen in the class! Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring.

If k is *not* algebraically closed then working with $\mathbb{A}_k^n = k^n$ does not give us the "right" theory; eg. $\mathbb{A}_{\mathbb{F}_2}^1 = \{0, 1\}$ has 'too few' points to reveal any geometry. To correct that we (re-)define the affine space, over *any field* k, as the set

$$\text{maxSpec A} := \{\text{M} \mid \text{M} \trianglelefteq \text{A is a maximal ideal}\}.$$

Develop the usual notions for this "new $\mathbb{A}_k^n$ " in the following steps.

(i) Show that $X :=$ maxSpec A can be made into a *Zariski topological space*. (I.e. re-define the closed/open subsets.) Is it reducible?

(ii) Let U be an open subset of X. How can we view an $f \in A$ as a map $U \to \overline{k}$. Define *regular functions* on U; denote the ring of regular functions by $\mathcal{O}_X(U)$.

(iii) Show that $\mathcal{O}_X(X) \cong A$.

(iv) Define *germs* at a point $M \in X$; denote the ring of germs by $\mathcal{O}_{X,M}$. How does it relate to A?

(v) Define *rational functions* on X; denote the ring of rational functions by $K(X)$. Show that $K(X)$ is a field. How does it relate to A?

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