On the Concept of Genus in Topology and Complex Analysis

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The English word "genus" hails from biology, where it is used to connote a grouping of organisms having common characteristics. In mathematics the word is also used to group objects with common characteristics. The concept of genus arises in various mathematical contexts, such as number theory, as well as in the areas we consider in this article, topology and complex analysis. Even within the latter two areas there are various notions of genus that historically originated with the genus of an oriented surface. We begin with these origins and afterward treat generalizations and modifications. We provide no detailed definitions and proofs; rather, our goal is to give the reader an intuitive feeling for the concept of genera.

The Genus of a Surface

In his paper "Theorie der Abel'schen Functionen" [20] Riemann studied the topology of surfaces. He classified a surface by looking for simple closed curves along which to cut in order to obtain a simple presentation of the surface. He called the minimal number of such curves $2p$ and showed that this invariant determines the surface. A few years later, when Clebsch studied surfaces from a more algebraic geometric viewpoint, he called p "*das Geschlecht (genus)*" of the surface.

In more modern terms one can formulate Riemann's insight as follows: every connected, closed (meaning compact without boundary), oriented surface F is obtained from the 2-sphere S^2 by taking repeated connected sums with the torus, $T = S^1 \times S^1$. The number of tori added is a topological invariant called the *genus of* F, $g(F)$, which equals Clebsch's p.

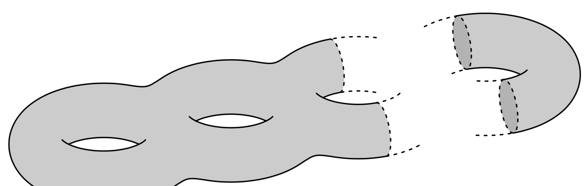

Figure 1.

Riemann's concept of surface is not easy to summarize. It is something like a ramified covering of the plane, and he implicitly assumes that a surface has a sort of differentiable structure, which is a great technical help. But the genus is actually a topological invariant. One can prove this by using either the fundamental group or the first homology group $H_1(F)$. Specifically, $H_1(S^2) = 0$, and the formula

$$H_1(F \sharp T) \cong H_1(F) \oplus \mathbb{Z}^2,$$

where $\sharp$ stands for connected sum, implies that a surface F_n obtained from S^2 by connected sum with n tori has

$$H_1(F_n) \cong \mathbb{Z}^{2n}.$$

The rank of the k-th homology group is called the k-th *Betti number*,

$$b_k(X) := \operatorname{rank}(H_k(X)),$$

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and so we obtain the formula

$$g(F) = b_1(F)/2.$$

Instead of the Betti number one can use the *Euler characteristic*,

$$e(X) := \sum_i (-1)^i b_i(X),$$

to determine the genus of F, namely, $b_0(F) = b_2(F) = 1$, and so $e(F) = 2 - b_1(F) = 2 - 2g(F)$, implying

$$g(F) = 1 - e(F)/2.$$

The Euler number itself can be computed combinatorially without referring to homology. Every surface has a *triangulation* (see next section), which we can visualize by putting a net of triangles over F

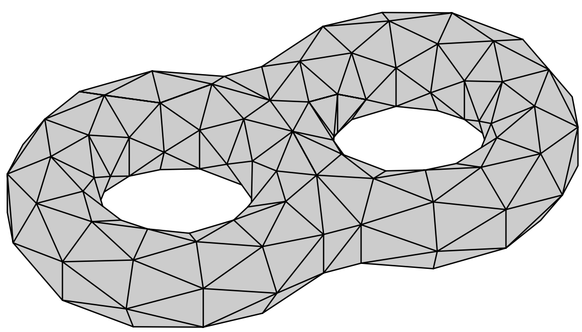


Figure 2.

and then

$$e(F) = \textit{number of vertices} - \textit{number of edges} + \textit{number of triangles}.$$

However, to show that the combinatorially defined Euler characteristic is a topological invariant needs a proof.

What Is the Topological Significance of the Genus?

So far we have attached a topological invariant, a certain characterization, to a surface: the genus. How powerful is this invariant? What is its topological significance? One can interpret Riemann as saying that a differentiable surface is determined by its genus. A modern proof of this can be obtained from elementary Morse theory [6], [7]. But a stronger statement is true: the genus characterizes the homeomorphism type. This result was proved by Rado long after Riemann's work:

Theorem [19]. *Two connected closed oriented surfaces F and F' are homeomorphic if and only if*

$$g(F) = g(F').$$

This is not an easy result. The central step in Rado's proof is to show that a topological surface with countable basis has a triangulation, and then a proof in the combinatorial world is not difficult.

What Is the Analytic Significance of the Genus?

The main intention of Riemann's topological considerations was to study surfaces as objects in complex analysis: in modern terms, as *complex manifolds* of complex dimension 1, *complex curves*. A complex manifold is a topological manifold (meaning a topological Hausdorff space with countable basis locally homeomorphic to $\mathbb{R}^n$) together with an atlas whose coordinate changes are holomorphic maps. We note that in complex dimension 1 a countable basis follows from the existence of a complex structure [19]. On such a complex curve Riemann considered divisors D, which are finite formal linear combinations of points in the surface with coefficients in $\mathbb{Z}$. Each meromorphic function on a closed surface (in the following we assume that the surfaces are closed) determines a divisor D by its zeroes and poles (counted with multiplicity). To each divisor we attach the sum of the coefficients called $\deg(D)$. Given D we can consider the vector space of meromorphic functions characterized by the property that its divisor plus the given divisor does not have negative multiplicities. This vector space is finite-dimensional, and its dimension is denoted by $l(D)$.

For a divisor D Riemann [20] proved his inequality

$$l(D) \geq deg(D) + 1 - g.$$

This is the starting point of the famous Riemann-Roch theorem [21], which gives an equality using the *canonical divisor* K:

$$l(D) - l(K - D) = deg(D) + 1 - g.$$

Why is this an important result? If we look at the right-hand side of the equation, we see the sum of a very simple invariant of a divisor, its degree, and a topological invariant, $1 - g$. However, the left-hand side is a complicated analytic invariant: the difference of dimensions of certain function spaces, more precisely spaces of meromorphic functions with restricted zeroes and poles.

The Arithmetic Genus of Algebraic Varieties

It should be noted that the passage from differentiable structures to complex structures is a dramatic change, since there are in general many different complex structures on a given surface. One has a moduli space of complex structures which for $g \geq 2$ is itself a complex manifold of complex dimension $3g - 3$. For $g = 0$ the moduli space is a point, and for $g = 1$ it has complex dimension 1.

One can proceed further and impose even more refined structures on a surface, for example, complex algebraic structures. Whereas topological and smooth manifolds were intensively studied in the first half of the last century, complex manifolds of dimension greater than 1 were not investigated very much by complex analytic methods (of course the function theory of several complex variables in open domains in $\mathbb{C}^n$ was a subject of great interest). In contrast, *algebraic varieties*, the set of zeroes of a family of polynomials, were the subject of constant mathematical investigations, at least by constructing interesting examples and studying their geometry. Many formulae were found that led to interesting questions and conjectures.

In this context another genus, the *arithmetic genus*, played an important role. In the early 1950s four definitions of the arithmetic genus of a projective smooth algebraic variety V of complex dimension n were known. The first two are denoted by $p_a(V)$ and $P_a(V)$. Severi conjectured that these numbers agree and can be computed in terms of the dimension $g_i(V)$ of the vector space of holomorphic differential forms of degree i:

$$p_a(V) = P_a(V) = g_n(V) - g_{n-1}(V) + \cdots + (-1)^{n-1} g_1(V).$$

The expression on the right-hand side is the third definition, which we recommend to the reader (actually in a slightly modified form described below). Using sheaf theory, Kodaira and Spencer [14] proved that the three expressions agree.

The expression on the right looks like an Euler characteristic, but in a strange form. The "correct" Euler number is the *holomorphic Euler number*,

$$\chi(V) := \sum_{i=0}^{n} (-1)^i g_i(V),$$

called the *arithmetic genus*. The number of components of V is $g_0(V)$. Thus for a connected variety $1 + (-1)^n p_a(V) = \chi(V)$. Often $g_n(V)$ is called the *geometric genus* of V. For the case of a curve (Riemannian surface) we have

$$g_1(V) = g(V),$$

and so the geometric genus and Riemann's genus agree.

Both the arithmetic and geometric genus are multiplicative:

$$\chi(V \times V') = \chi(V)\chi(V')$$

and

$$g_{n+m}(V \times V') = g_n(V) g_m(V'),$$

where $n = \dim V$ and $m = \dim V'$.

The $g_i(V)$ are birational invariants [25], and so the arithmetic genus is a birational invariant.

The Todd Genus

The fourth definition of the arithmetic genus was given by J. A. Todd [24]. A canonical divisor of a smooth projective algebraic variety of dimension n is given as a divisor of a meromorphic n-form. It is an algebraic cycle of topological codimension 2. Todd introduced geometric canonical cycles for all even codimensions. He defined polynomials in these cycles, where the product is given by intersections. The n-th Todd polynomial is of codimension $2n$ and represents for a variety of dimension n a certain number called the *Todd genus*. Todd believed that his genus was the same as the arithmetic genus, but rigorous justification of this fact came much later.

The Todd canonical classes represent homology classes, and they are up to signs Poincaré dual to the *Chern classes* of the tangent bundle of the variety [18]. More generally, Chern classes are defined for complex vector bundles. In contrast to complex manifolds, where holomorphic maps play a definitive role, complex vector bundles are purely topological objects; roughly speaking, they are a family of complex k-dimensional vector spaces parametrized by the points of a topological space X. There is also a topology on the disjoint union of these vector spaces, and the key property is that locally this family of vector spaces is homeomorphic to a product $U \times \mathbb{C}^k$, where U is an open subset of X. The formal definition of Chern classes is too complicated for an article such as this one, but the basic idea behind them can be explained by considering the case of a differentiable complex vector bundle E over a closed differentiable manifold X. Then the Chern class $c_i(E) \in H^{2i}(X;\mathbb{Z})$ is the first obstruction to the existence of $k - i + 1$ linearly independent sections on E. If one chooses, for example, a single section on E and the choice is generic, then the set of zeroes is a submanifold of X of dimension $\dim X - 2 \dim E$. Thus we obtain a homology class whose Poincaré dual sits in $H^{2k}(X)$ and is the k-th Chern class $c_k(E)$ of E. If X is a closed complex k-dimensional manifold and E its complex tangent bundle, then $c_k(E)$ evaluated on the fundamental cycle is the Euler characteristic of X by the Poincaré-Hopf theorem.

The Todd genus in terms of the Chern classes of the tangent bundle is the evaluation of a certain rational polynomial in the Chern classes $T(c_1, c_2, \dots)$ on the fundamental class. To motivate the construction of these polynomials (which are rather complicated expressions), we note that if the Todd genus agrees with the arithmetic genus, then certainly for complex projective spaces (where the arithmetic genus takes the value 1) they have to agree. Furthermore, the Todd genus has to be multiplicative in a way that reflects the multiplicativity of the arithmetic genus, and so it should be a multiplicative sequence in the sense of [8]. This was the motivation for the first author to introduce the general concept of multiplicative sequences of polynomials. He characterizes the Todd sequence by a special multiplicative sequence with value 1 on each complex projective space.

The first four polynomials are

$$T_1 := \frac{1}{2}c_1,$$

$$T_2 = \frac{1}{12}(c_1^2 + c_2),$$

$$T_3 = \frac{1}{24}c_1c_2,$$

$$T_4 = \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4).$$

Theorem [8]. *Let V be a nonsingular compact complex algebraic variety of dimension n. Then*

$$\chi(V) = \langle T_n(V), [V]\rangle,$$

the evaluation of the Todd polynomial T_n on the fundamental class.

We see that this result has a similar flavor to the Riemann-Roch Theorem, since it relates analytic information, the holomorphic Euler number, to a topologically defined invariant, the Todd genus.

If the complex dimension of V is 1, the case of a Riemannian surface, the theorem above is a special case of the Riemann-Roch formula above, namely, the case where $D = 0$. In the same sense, the theorem above is the special case of the Hirzebruch-Riemann-Roch formula for $D = 0$ [8]. In 1957 Grothendieck generalized this by considering a parametrized version of the Riemann-Roch formula [2]. All this is a long story, and although closely related to genera, it would lead us too far away from our main themes.

Bordism and Generalized Genera

When we said that the Todd genus is topologically defined, this was too brief. In addition to the underlying differentiable manifold, one needs a complex structure in a weaker sense: *a complex structure on the sum of the tangent bundle with a trivial bundle.* This structure is called a stable almost complex structure. A manifold with a stable almost complex structure is called a stable almost complex manifold. Note that with this definition, even an odd-dimensional manifold can have a stable almost complex structure.

Since the Chern classes are stable invariants, which means that they are unchanged if we add a trivial (complex) bundle, the structure one needs for defining the Todd genus is a stable almost complex structure. For such manifolds the Todd genus has the following fundamental properties:

- it is additive (i.e., the Todd genus of a disjoint union is the sum of the Todd genera)
- it is multiplicative (i.e., the Todd genus of a product is the product of the Todd genera).

These properties of the Todd genus motivated the first author to introduce the general concept of *genus*. This is an invariant Φ for certain classes of manifolds in terms of characteristic classes of the tangent bundle (perhaps equipped with a stable almost complex structure) with values in a ring Λ fulfilling the two properties above. We note that for a Riemannian surface F the Todd genus is $c_1(F)/2$, which is half the Euler characteristic of F, namely, $1 - g(F)$. Thus the Todd genus is in this case essentially the genus of a Riemannian surface.

An important invariant of oriented manifolds is the *signature* $(b_+ - b_-)$, which is the signature in the sense of linear algebra of the intersection form of a $4k$-dimensional closed oriented manifold M (if the dimension is not divisible by 4, the signature is defined as zero). It is denoted by

$$sign(M) \in \mathbb{Z}.$$

The first author was looking for a formula that, in analogy to the formula for the arithmetic genus, computes the signature in terms of characteristic classes. This was done at a time when the Riemann-Roch formula was only conjectured. In fact, the signature theorem became an important ingredient in the proof of the Riemann-Roch formula. Since there is no complex structure on the tangent bundle, one has to use the Pontrjagin classes $p_i(M) \in H^{4i}(M)$ instead of the Chern classes. These are (up to sign) the Chern classes of the complexification of the tangent bundle. In analogy to the Todd genus, the first author used his formalism of multiplicative sequences to construct polynomials in Pontrjagin classes that take for each even-dimensional complex projective space the value 1, the signature of M. These are the L-polynomials. The first three L-polynomials are

$$L_1 = \frac{1}{3}p_1,$$

$$L_2 = \frac{1}{45}(7p_2 - p_1^2),$$

$$L_3 = \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3).$$

If, as for the arithmetic genus, one knew that the values on the even-dimensional projective spaces characterize the signature, one would obtain the desired formula. This would follow if, after passing to a multiple if necessary, each manifold were bordant to a linear combination of products of projective spaces. The reason is that both the signature and the L-polynomials are bordism invariants. Here two oriented manifolds M and N are bordant if there is a compact oriented manifold W with boundary being the disjoint union of M and $-N$, the manifold N with the opposite orientation.

The bordism classes of closed oriented n-dimensional manifolds form a group under disjoint union, denoted by Ω_n. The sum

$\Omega_* := \sum_n \Omega_n$ is a ring with respect to the product of two manifolds. Thom [23] computed $\Omega_* \otimes \mathbb{Q}$. It is the polynomial ring with generators, the even-dimensional projective spaces. These considerations lead to:

Theorem *(Signature Theorem)* [8]. *Let M be a closed smooth oriented manifold. Then*

$$sign(M) = \langle L(M), [M] \rangle.$$

Returning to the Todd genus, we noted that it is defined for closed manifolds with stable almost complex structure. In analogy to the bordism groups of oriented manifolds, Milnor [16] defined and computed bordism groups of stable almost complex manifolds. The answer is simpler than for oriented bordism groups: the bordism ring of stable almost complex manifolds is a polynomial ring over $\mathbb{Z}$ in variables x_i corresponding to stable almost complex manifolds of real dimension $2i$, which Milnor explicitly describes. If one takes the tensor product with $\mathbb{Q}$, generators are given by the projective spaces $\mathbb{CP}^i$.

The Relevance of the Signature

Here we can only indicate some aspects (in a nonhistorical order). The classical genus of a Riemannian surface completely characterizes the homeomorphism type (which for surfaces agrees with the diffeomorphism type). In a certain sense one has an analogous result for closed smooth simply connected 4-manifolds.

Theorem [5], [3]. *Two closed differentiable simply connected 4-manifolds are homeomorphic if and only if the Euler characteristic, the signature, and the type (even or odd) agree.*

Here the type is even if and only if all self-intersection numbers are even. This is a very deep result based on independent difficult theorems by Freedman and Donaldson. Freedman classified simply connected topological 4-manifolds in terms of the intersection form and a $\mathbb{Z}/2$-valued invariant, the Kirby-Siebenmann invariant. This vanishes for smooth manifolds as well as for manifolds homotopy equivalent to S^4. Thus, as a special case, Freedman proves the topological 4-dimensional Poincaré conjecture: A 4-dimensional manifold homotopy equivalent to S^4 is homeomorphic to S^4. Every unimodular symmetric bilinear form is the intersection form of a closed simply connected topological 4-manifold, and the classification of such forms is unknown. But for smooth manifolds Donaldson used gauge theory to show that the intersection forms are very special and, because of some classical results, are classified by the rank (equivalent to the Euler characteristic), the signature, and the type.

In contrast to Riemann surfaces, the analogous result for a diffeomorphism classification is completely different in dimension 4. There are many simply connected 4-manifolds M that have an exotic smooth structure, which means there exists another manifold homeomorphic but not diffeomorphic to M. The first example was found by Donaldson [4]. Later on, his techniques were applied to show that many simply connected 4-manifolds have infinitely many smooth structures, for example, the $K3$-surface $\{x \in \mathbb{CP}^3 \mid \sum x_i^4 = 0\}$ (a complex surface, so the real dimension is 4).

The following is one of the big open problems in differential topology: Is there any closed 4-manifold with no exotic structure? The most interesting examples would be the complex projective plane $\mathbb{CP}^2$ or the 4-sphere S^4. If S^4 has a unique smooth structure, this is the smooth 4-dimensional Poincaré conjecture, which one can formulate as follows: A closed smooth simply connected 4-manifold with Euler characteristic 2 is diffeomorphic to S^4 (it has automatically second Betti number 0 and so signature 0; thus by the theorem above it is homeomorphic to S^4).

The existence of infinitely many smooth structures on a closed manifold is something that happens exclusively in dimension 4. In all other dimensions this number is finite. This result is closely related to the Hauptvermutung, which says that if a topological manifold can be triangulated, then this triangulation is unique up to refinement. This is not true (the first counterexamples were given by Milnor [17]), but in dimension > 4 the work of Kirby and Siebenmann about the Hauptvermutung [13] shows that a topological manifold of dimension > 4 has at most finitely many piecewise linear structures. Using surgery theory, one can show that a piecewise linear manifold of dimension > 4 has at most finitely many smooth structures. Combining these two results, one sees that a topological manifold of dimension > 4 has at most finitely many smooth structures.

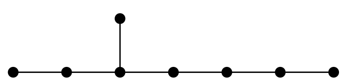
In this last result the classification of smooth structures on spheres plays an essential role (for the first examples by Milnor, see [15]; for the general classification in dimension > 4 by Kervaire and Milnor, see [12]). The signature and in particular the signature theorem are as much central tools for the existence of exotic structures on spheres as tools for the classification of such structures. We indicate this for existence. Milnor constructs certain compact smooth manifolds W with boundary homeomorphic to S^{4n-1}. Then he considers the union of W with the cone over its boundary. If the boundary is diffeomorphic to S^{4n-1}, then this is a smooth manifold, and thus one can compute its signature by the signature theorem in terms of the L-polynomials. Except for the expression in the top Pontrjagin class p_n, all other terms in the L-polynomial can be computed in terms of the Pontrjagin classes of W. Thus one can use the signature formula to compute the term in p_n

in the L-polynomial. The coefficient of p_n in the L-polynomial is a rational number,

$$\frac{2^{2n}(2^{2n-1}-1)}{2n!}(-1)^{n-1}b_{2n},$$

where b_{2n} is the Bernoulli number. If the boundary of W is diffeomorphic to S^{4n-1}, the fact that p_n is an integer gives a certain congruence between the difference of the signature of the union of W with the cone over the boundary and the other expressions in the L-polynomial (we will carry out an especially simple example). If this congruence does not hold, the boundary of W is not diffeomorphic to the sphere. This way one can produce examples of exotic structures on spheres of dimension $4n-1$ for $n > 1$ (see also [17]).

We want to use plumbing [9] similar to a construction used by Milnor to give explicit examples of exotic spheres and at the same time the construction of a topological manifold without smooth structure (the existence of such manifolds was first shown by Kervaire [11]). We consider the E_8-graph



Using E_8, we construct a manifold with boundary of dimension 12 by gluing together for each edge a copy of the disc bundle of the tangent bundle of S^6 using the following recipe. If two vertices are joined by an edge, we take a trivialization of the disc bundle over a disc D^6 in S^6 to obtain an embedding of $D^6 \times D^6$ into the disc bundle, where the first component maps to S^6 and the second to the fibres. Then we identify (x, y) in the first product with (y, x) in the second:

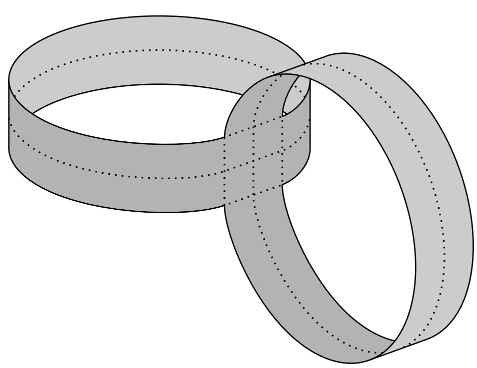


Figure 3.

The result is a compact 12-dimensional manifold $W(E_8)$ with boundary and corners, which one can smooth. By construction it is homotopy equivalent to a wedge of eight copies of S^6. By general position the fundamental group is trivial. Using the unimodularity (meaning that the intersection form has determinant ± 1) of the E_8-form, we find that the Mayer-Vietoris sequence implies that the boundary of $W(E_8)$ is a homotopy sphere. Application of the Poincaré conjecture proved by Smale [22] shows that the boundary is homeomorphic to S^{11}. We now apply the signature theorem to show that it is not diffeomorphic to S^{11} following the principle explained above. If there were a diffeomorphism $f : \partial W(E_8) \to S^{11}$, then we could consider $M := W(E_8) \cup_f D^{12}$ and obtain a smooth manifold whose homology is trivial except in degree 0, 6, and 12. The intersection form of this manifold is by construction the E_8-form, whose signature is 8. Now we apply the signature theorem and obtain

$$8 = \frac{62}{945}\langle p_3(M), [M]\rangle,$$

a contradiction (note that the only potentially nontrivial Pontrjagin class is $p_3(M)$, an *integral* cohomology class). Thus $\partial W(E_8)$ is an exotic sphere. If instead of a diffeomorphism we use a homeomorphism, we obtain a topological manifold M, which by the same argument as above cannot admit a smooth structure.

The use of the E_8-graph is motivated by the fact that the resolution of the singularity in $(0,0,0)$ of $z_1^2 + z_2^3 + z_3^5 = 0$ consists of eight nonsingular rational curves of self-intersection number -2 whose intersection behavior is given by E_8.

The Atiyah-Singer Index Theorem and Other Genera

Except for the signature, the left-hand sides of our formulas related to the genus were of an analytic nature, being given by dimensions of certain vector spaces of functions or differential forms. In fact the signature can also be interpreted as an analytic invariant via Hodge theory. It is the index of a differential operator, the *signature operator*. Thus the signature theorem is an index theorem expressing the index of an elliptic differential operator in topological terms.

During the 1960s Atiyah and Singer [1] proved a general index theorem for elliptic differential operators on smooth manifolds extending for example the signature theorem. Besides the signature the most important operators are the Laplace operator whose index is the Euler characteristic and the Dirac operator on a manifold with spin-structure. The topological side of the index formula for the Dirac operator, the $\hat{A}$-genus, was studied in [8]. In the 1980s Ochanine and Witten defined very interesting genera for spin, respectively string, manifolds, the *Ochanine*, respectively *Witten, genus*. The remarkable property of these genera is that they take values in rings of modular forms (compare, for example, [10]). One can conjecture that these genera are only a shadow of interesting new (co)homology theories associated with the term *elliptic cohomology*. In the end one expects that elliptic cohomology will play a central role in index theory on the loop space of a manifold in analogy with the role that K-theory plays in index theory on smooth manifolds.

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Noticed

Medicine and Mathematical Humor

Citations to *Notices* articles crop up in many places—but in a medical journal? In 2006, *Best Practice & Research Clinical Obstetrics and Gynaecology* carried an article called "Assessing new interventions in women's health". Written by University of Birmingham biostatistician Robert K. Hills and research fellow Jane Daniels, the article discusses principles for running or assessing clinical trials of medical treatments for women, with an emphasis on how to evaluate the treatments' efficacy in the face of unclear or conflicting trial results. "[I]t is not acceptable to rely on proof by anecdotal evidence, eminent authority, or vigorous handwaving," the authors write, citing the widely read *Notices* article "Foolproof: A sampling of mathematical folk humor", by Alan Dundes and Paul Renteln (January 2005). As he later explained to *Notices* Editor Andy Magid, Hills was an undergraduate mathematics student at the University of California, Los Angeles, and heard jokes about the various methods of proof (such as "proof by intimidation"), many of which are listed in the Dundes-Renteln piece. Hills and Daniels also quote Bertrand Russell: "The fact that an opinion has been widely held is no evidence whatever that it is not utterly absurd."