CS688 - COMPUTATIONAL ARITHMETIC-GEOMETRY & APPLICATIONS

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END-SEMESTER EXAMINATION

POINTS: 65

DATED: 27-APR-2023 DUE: 01-MAY-2023 (10AM)

Rules:

- You are strongly encouraged to work *independently*. You are not allowed to discuss.
- Write the solutions on your own and honorably *acknowledge* the sources if any. `cse.iitk.ac.in/pages/AntiCheatingPolicy.html`
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Submit your solutions, before time, to your TA. Preferably, submit a printed/pdf copy of your LaTeXed or Word processed solution sheet.
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Question 1: [*3+3 points*] Show that the smoothness of curves is not preserved under birational equivalence.

- How does the number of $\mathbb{F}_q$-points on a curve change when we move to its smooth model?

Question 2: (Genus one) [*9+5 points*] Let C be a genus one smooth projective curve over k (any field) and say it has a k-point P. Show that C has an affine description $y^2 + h(x)y = f(x)$, for some $h, f \in k[x]$ of degrees at most $1, 3$ respectively.

[*Hint*: Can you utilize $L(iP)$?]

- Finally show that any such C, over a field where $6 \neq 0$, has an affine description $y^2 = x^3 + ax + b$, with $4a^3 + 27b^2 \neq 0$.

Question 3: (Fermat curve) [*18 points*] Compute the genus of the curve $C : x_0^n + x_1^n + x_2^n = 0$ over any algebraically closed field k.

Question 4: [*5+6 points*] Let C be a smooth projective curve, over $k = \mathbb{F}_q$, of genus g. Let $J(C)$ be its Jacobian variety. Estimate the size $|J(C)|$.

- What can you say about the structure of the group $J(C)$ as a product of cyclic groups?

Question 5: [*4+4+8 points*] Let k be a field where $6 \neq 0$.

Consider the affine curve $C : y^2 = x^3 - x$ and its projective version $\tilde{C} : y^2 z = x^3 - xz^2$. Show that $\tilde{C}$ is non-singular at *all* points.

- Now, consider $C : y^2 = x^4 - x$ and its projective version $\tilde{C} : y^2 z^2 = x^4 - xz^3$. Show that $\tilde{C}$ has a singularity. Thus, plain *homogenization* does not always give a smooth model for varieties!

- Describe a smooth model of the curve $y^2 = x^4 - x$.

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