CS688 - COMPUTATIONAL ARITHMETIC-GEOMETRY & APPLICATIONS
NITIN SAXENA

ASSIGNMENT 3

POINTS: 70

DATE GIVEN: 13-MAR-2023 DUE: 02-APR-2023 (6PM)

Rules:
- You are strongly encouraged to work *independently*. That is the best way to understand the subject.
- Write the solutions on your own and honorably *acknowledge* the sources if any. `cse.iitk.ac.in/pages/AntiCheatingPolicy.html`
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Submit your solutions, before time, to your TA. Preferably, submit a printed/pdf copy of your LaTeXed or Word processed solution sheet.
  TA: Diptajit Roy `diptajit@cse.iitk.ac.in`
- Problems marked '0 points' are for practice.

Question 1: *[6 points]* Let C be a smooth projective curve, over a field k (any). Let P be a point in C. Show that there is always a rational function, in $K(C)$, whose only pole is P.

Let k be an algebraically closed field for Qns.2-4.

Question 2: *[4+3 points]* Let C be the projective line $\mathbb{P}^1_k$. Show that
- $\mathrm{Cl}_0(C) = \{0\}$.
- $\mathrm{Cl}(C) = \mathbb{Z}$.

Question 3: *[7 points]* Show that the projective line has genus zero.
(Hint: Compute $\ell(nP)$.)

Let's now prove the converse.

Question 4: (Immersion) *[10+8+7 points]* Let C be a curve with genus g. Let $D \in \mathrm{Div}(C)$, and let $s_0, \ldots, s_n \in L(D)$ be a k-basis. Show that

(i) If $d(D) > 2g - 1$, then $\varphi = (s_0, \ldots, s_n) : C \to \mathbb{P}^n$ is a morphism.

(ii) If $d(D) > 2g$, then φ is injective.

- Deduce that if C has genus zero, then it is the projective line.

Question 5: *[7 points]* Let D be a divisor of a curve C, with the canonical divisor being W and genus g. If $d(D) = 2g-2$ and $\ell(D) = g$, then $D = W$ (in $\mathrm{Cl}(C)$).

Question 6: (Clifford) *[12+6 points]* Let D be a positive divisor of a curve C, with the canonical divisor being W. If $\ell(W - D) > 0$, then prove

(i) $\ell(D) + \ell(W - D) \leq \ell(W) + 1$.

(ii) $\ell(D) \leq d(D)/2 + 1$.

Question 7: *[0 points]* Show that an equality in (ii) above, implies: $D = 0$, or D is canonical, or $\exists D'$ satisfying $\ell(D') = d(D') = 2$.

(Remark: In the last case, C is called *hyperelliptic* and D' provides a degree two morphism $C \to \mathbb{P}^1$.)

Question 8: *[0 points]* Let D, D' be divisors of a curve, with Ω being the differential space. Show that $\Omega(D) \cap \Omega(D') = \Omega(\mathrm{lcm}(D, D'))$.

Question 9: *[0 points]* Let C be a smooth projective curve, over an algebraically closed field k, of genus *one*. Give a characterization of this curve using Riemann-Roch.

Question 10: *[0 points]* Show that the (algebraic) definition of genus given in the class is consistent with the definitions of genus given in complex analysis (eg. number of 'holes' in an orientable surface).

Question 11: *[0 points]* Complete the linear-algebraic proof of the following property of the L-sheaf done in the class, given divisors $D \leq D'$ and $S := \mathrm{supp}(D') \cup \mathrm{supp}(D)$:

$$L(D')/L(D) \;\cong\; L(D')_S/L(D)_S \,.$$

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