CS688 - COMPUTATIONAL ARITHMETIC-GEOMETRY & APPLICATIONS

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ASSIGNMENT 2

POINTS: 50

DATE GIVEN: 08-FEB-2023 DUE: 19-FEB-2023 (6PM)

Rules:

- You are strongly encouraged to work *independently*. That is the best way to understand the subject.
- Write the solutions on your own and honorably *acknowledge* the sources if any. `cse.iitk.ac.in/pages/AntiCheatingPolicy.html`
- Clearly express the fundamental *idea* of your proof/ algorithm before going into the other proof details. The distribution of partial marks is according to the proof steps.
- There will be a penalty if you write unnecessary or unrelated details in your solution. Also, do not repeat the proofs done in the class.
- Submit your solutions, before time, to your TA. Preferably, submit a printed/pdf copy of your LaTeXed or Word processed solution sheet.
  TA: Diptajit Roy `diptajit@cse.iitk.ac.in`
- Problems marked '0 points' are for practice.

Question 1: *[7 point]* Show that the radical of a homogeneous ideal is again homogeneous.

Question 2: *[2+6 points]* What is a *reducible* topological space?
Discuss this notion (with proofs) in the context of the Euclidean *real line* versus the *affine line* (i.e. $\mathbb{A}_k^1$)?

Question 3: *[4+2+5 points]* Recall *morphisms* and *rational maps* between varieties. Give an example of varieties that are not isomorphic wrt the former but are isomorphic wrt the latter notion. Is the opposite case possible?
- Show that the morphisms $X \to Y$ between affine varieties are in 1-1 correspondence to the k-algebra homomorphisms $A(Y) \to A(X)$.

- Let $\varphi : p \mapsto (f_1(p), \ldots, f_n(p))$ be an automorphism (wrt morphisms) of $\mathbb{A}^n_{\mathbb{C}}$. Show that the Jacobian $\det((\partial_{\rm i} {\rm f}_{\rm j}))$ is a nonzero constant (in $\mathbb{C}$).

Question 4: (Product) *[4+3+4 points]* Show that for affine varieties X, Y the set product $X \times Y$ can also be viewed as an affine variety.

- (Segre embedding) Generalize the product to projective varieties.
- How does the (geometric) dimension grow with these products?

Question 5: *[3+3+3+4 points]* Consider the projective Fermat curve $X := Z(x_1^n + x_2^n = x_3^n) \subset \mathbb{P}^2_{\mathbb{C}}$.

- Compute $\mathcal{O}_X(x_3 \neq 0)$.
- Consider any point $P \in X$. Compute the germs $\mathcal{M}_P$ and $\mathcal{O}_{X,P}$.
- Compute $\mathcal{M}_P/\mathcal{M}_P^2$ and $T_{X,P}$.
- Is X a nonsingular projective curve?

Question 6: *[0 point]* Show that dimension is a *local* concept, i.e. for any point P in a variety X, $\dim X = \dim \mathcal{O}_{X,P}$.

Question 7: *[0 point]* Let $k \subset K$ be a trdeg one field extension, $k \subseteq R$ be a *subring* of K, and $\mathcal{M}_R \trianglelefteq R$ be some maximal ideal. Show that $(\mathcal{M}_R, R)$ extends to a dominating dvr $(\mathcal{M}_B, B)$.

- How many possibilities are there for B?

Question 8: *[0 point]* Show that any nonsingular affine curve X can be embedded in some nonsingular *projective* curve Y.

Question 9: *[0 point]* Given a curve X, are there fast algorithms:

- To find all singular points on X?
- To find a nonsingular model of X?

Question 10: *[0 point]* Let $k \subset K$ be a trdeg $= 1$ field extension. Compute the ring $\bigcap\{R_v \mid \text{valuation } v \text{ of } K\}$.

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