CGS600A: Computational Tools in Cognitive Science

Assignment #6: Elementary probability

Max marks:150

Due on/before:29 Oct.18, 23.59 22-Oct.-2018

1. (a) Let A, B, C be 3 events. Using the laws of probability and any derived results find the probability for $P(A \cup B \cup C)$.

   (b) Using the above calculation as a base find the formula to calculate the probability of $P(E_1 \cup E_2 \cup \dots E_n)$ where E_1 to E_n are n events.

   (c) A fair 6 sided die is tossed. Let its outcome be denoted by X. Then a fair coin is tossed X times and Y denotes the number of heads in X tosses.

   Calculate the following:

   1. $P(Y = 4)$
   2. $P(X = 5|Y = 4)$
   3. $E(Y)$
   4. $E(XY)$

   (d) Cards with numbers 1 to 10 written on them are put into a hat and shuffled thoroughly. One card is drawn and you are told that the number on it is at least 5. Calculate the probability that the number on the card is 10.

   (e) A family has 2 children. Find the conditional probability that both are girls given that at least one of them is a girl. Assume $S = \{(b, g), (g, b), (g, g), (b, b)\}$ where b - stands for boy and g - for girl and (g, b) means the first child is a girl and the second a boy. Assume each outcome has the same probability.

   (f) Let a box contain 2 white and 7 black marbles and another box contain 5 white and 6 black marbles. A fair coin is flipped. If its heads a marble is drawn from the first box else from the second box.

   Calculate the conditional probability that the coin toss was heads given that the marble drawn was white.

   (g) In a multiple choice test with m alternatives per question the probability that a guess would get the correct answer is $\frac{1}{m}$. The probability that the student knows the correct answer is p, so $1 - p$ is the probability the student will guess the answer.

   Calculate the probability that the student knew the answer to the question given that it was answered correctly.

   (h) This probability calculation is important in practice. A lab test for a disease is accurate 95% when a person actually has the disease. It also gives a false positive 1% of the time - that is is positive when the person does not have the disease.

   Assuming that the incidence of the disease in the population is 0.5% what is the probability that a person has the disease given that the test result is positive.

(i) If E_1 to E_n are events argue that $P(\cup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} P(E_i)$. When will equality hold? This is called Boole's inequality.

(j) If E_1 to E_n are events argue that $P(E_1 E_2 \ldots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\ldots P(E_n|E_1\ldots E_{n-1})$.

(k) Assume n ladies throw their handkerchiefs into a box that is then shaken thoroughly. Each then randomly draws a handkerchief from the box. Calculate the probability that none of the ladies draws her own handkerchief. What happens when $n \to \infty$?

(l) If E is the expectation operator, X a random variable and a, b constants argue that $E(aX+b) = aE(X)+b$.

(m) If E_1 to E_n are random variables and a_1 to a_n are constants then $E(\sum_{i=1}^{n} a_i E_i) = \sum_{i=1}^{n} a_i E(E_i)$ where $E(.)$ is the expectation operator. This is called the **linearity of expectation** and is very useful property. Argue for the above result.

(n) Calculate the expectation of random variable X where X has a) a Bernoulli distribution b) a Binomial distribution and c) a Poisson distribution. The distributions are:

Bernoulli: A Bernoulli experiment has only two outcomes success (1) and failure (0) and the distribution is: $P(X=1)=p$, $P(X=0)=1-p$.
Binomial distrubtion: A Bernoulli experiment is repeated n times and the probability of i success is: $P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$, $i = 0 \ldots n$.
Poisson: X has a Poisson distribution with parameter $\lambda > 0$ if $P(X=i) = e^{-\lambda}\frac{\lambda^i}{i!}$.

(o) Find the variances of random variable X distributed as in the previous question.

[10×15=150]