Communicative Approximations as Rough Sets

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Abstract. Communicative approximations, as used in language, are equivalence relations that partition a continuum, as opposed to observational approximations on the continuum. While the latter can be addressed using tolerance interval approximations on interval algebra, new constructs are necessary for considering the former, including the notion of a "rough interval", which is the indiscernibility region for an event described in language, and "rough points" for quantities and moments. We develop the set of qualitative relations for points and intervals in this "communicative approximation space", and relate them to existing relations in exact and tolerance-interval formalisms. We also discuss the nature of the resulting algebra.

1 Tolerances for Points and Intervals

When telling someone the time, saying "quarter past nine" has an implicit tolerance of about fifteen minutes, whereas the answer "9:24" would indicate a resolution of about a minute. Communication about quantities are defined on a shared conventional space, which constitutes a tesselation on the real number line. In this paper, we attempt to develop the first steps toward a theory that formulates these questions in terms of an indistinguishability relation [11], defining a tolerance approximation space common to participants in the discourse.

We take the *communicative approximation space* to be a set of cultural conventions that define a hierarchy of tesselations on a continuum. The two statements above reflect differing tolerances, defined on different discrete tesselations (granularities). The granularity adopted in a speech act reflects the measurement error or task requirement, and typically adopts the closest tesselation available in the shared communicative approximation space.

The communicative approximation space $\mathbb{C}$ then gives a discourse grid that is available at a number of scales, defined by equivalence classes (e.g. of one

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minute, five minutes, quarter hour, etc.). Similar questions of scale also inform communicative models for other continua such as space or measures.

The uncertainty resulting from measurement error $\pm\eta$ is defined on a tolerance approximation space $\mathbb{T}$ defined on a continuum, whereas the uncertainty reflected in communication, say ϵ, is defined on a discrete set of scales defined in the communicative approximation space $\mathbb{C}$. The ϵ-tesselation is a partition of the space via a series of ticks or *grid points*. These partitions or intervals then represent equivalence classes underlying the utterance. In honest communication, given a hierarchy of tesselations, ϵ will be chosen so as to be less precise than $\pm\eta$, in order to avoid a false impression of greater precision. Thus, we may assume that $\epsilon \geq |\eta|$. The greatest flexibility (worst case) arises when $\pm\eta = \epsilon$, and this is what we shall be assuming in the rest of this paper.

1.1 Measurement Tolerance vs. Communication Succinctness

The mapping from quantitative measures to common conceptual measures involves a step-discretization which has been the subject of considerable work in *measure theory* [5,15], *mereotopology* [2,16] and *interval analysis* [17,1]. At the same time, there is a rich tradition on information granulation in *rough set theory* [9,12]. Pawlak's premise [11] was that knowledge is based on the ability to classify objects, and by object one could mean 'anything we can think of' – real things, states, abstract concepts, processes, moments of time, etc. The original mathematical formulation of this assumption was manifested in the notion of an "approximation space": the domain of discourse, together with an equivalence relation on it.

$\mathbb{R}^+$, the set of non-negative real numbers, partitioned by half-open intervals [i,i+1), i=0,1,2,..., is an approximation space that is relevant to our work. One may remark that, in [9], Pawlak defines "internal" and "external measures" of any open interval (0,r) based on this partition, giving rise to a "measurement system". Later, in [8], this "inexactness" of measurement is further discussed, and contrasted with the theory of measurement of [14].

Our approach is related to *interval algebra* and *qualitative reasoning* [1]; however these operate with exact intervals and ignore tolerances. In this work, we develop the idea of interval tolerances [7] and map these onto communicative space discretization. We restrict ourselves to intervals, defined with two endpoints, with a single uncertainty ϵ. The next sections introduce the notion of a "rough interval" defined in terms of lower and upper approximations on the ϵ-tesselation. The end points of these rough intervals are indiscernibility regions which we call "rough points" by analogy to the continuum situation, though these are not rough sets except in a degenerate sense, since the lower approximation is empty. This extends the rough set [10] characterization for moments. Qualitative relations for rough points and intervals are defined, and compared with existing relations in the tolerance interval framework. A preliminary study is made of the relational algebraic structures that result from these constructs.

2 Rough Point

We consider a discretization of the continuum $\mathbb{R}$ by a (granularity) measure ϵ ($\in \mathbb{R}^+$). A real quantity ζ is taken as a 'reference point'. The communicative approximation space $\mathbb{C}$ is then a partition on $\mathbb{R}$ with the half-open intervals $[\zeta + k\epsilon, \zeta + (k+1)\epsilon)$, k being any integer. The points $\zeta + k\epsilon$ are called **grid points** (or 'ticks'), and the collection of grid points is called the **grid space**. Each grid interval is equivalent to an ϵ measure in $\mathbb{R}$. We notice that $\mathbb{C}$ is an approximation space that is a generalization of the one considered by Pawlak in [9] (cf. Section 1.1).

Note 1. For simplicity, we denote the k-th grid point, viz. $\zeta + k\epsilon$, as k, and the communicative approximation space $\mathbb{C}$ is taken to be the continuum $\mathbb{R}$ with this simplified representation of the discretization.

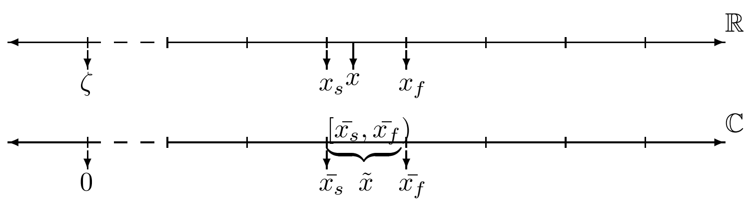
Observation 1. *The grid space is isomorphic to the set of integers, $\mathcal{Z}$.*

Example 1. A discretization of $\mathbb{R}$ with $\epsilon = 0.5,\ \zeta = 1.2$, would map a real line with grid points at 0.2, 0.7, 1.2, 1.7, 2.2, ... The communicative space then has the interval [0.2,0.7) as "-2", [0.7,1.2) as "-1", [1.2, 1.7) as "0" etc.

Now, given an exact x, one can locate the interval $[x_s, x_f)$ of $\mathbb{R}$ in which x lies (x_s, x_f are the 'start', 'end' of the interval). Thus $x_s = \zeta + \epsilon\bar{x_s},\ x_f = \zeta + \epsilon\bar{x_f}$, where $\bar{x_s} \equiv \lfloor \frac{x-\zeta}{\epsilon} \rfloor$, and $\bar{x_f} \equiv \lfloor \frac{x-\zeta}{\epsilon} + 1 \rfloor$. $[\bar{x_s}, \bar{x_f})$ is the corresponding interval in $\mathbb{C}$. Note that $\bar{x_f} = \bar{x_s} + 1$.
In the above example, the real number $x = 0.9$ would lie in the interval $[-1, 0)$ or, equivalently, in the interval $[0.7, 1.2)$ of $\mathbb{R}$.

Definition 1. *A* ***rough point*** *is any interval* $[k, k+1)$ *in* $\mathbb{C}$.

We observe that $[k, k+1)$ is a representation in $\mathbb{C}$ of all such real points x in $[\bar{x_s}, \bar{x_f})$, and *we denote it as* $\tilde{x}$. In other words, $\tilde{x}$ is the denotation for the unique equivalence class in $\mathbb{C}$ of $x \in [k, k+1)$. The quotient set $\mathbb{R}/\epsilon$ is thus the collection of all rough points.



For any rough point $\tilde{x}$ in $\mathbb{C}$, $\tilde{x} + 1_\epsilon$ and $\tilde{x} - 1_\epsilon$ are defined respectively as:

$$\tilde{x} + 1_\epsilon \equiv [\bar{x_s} + 1, \bar{x_f} + 1),$$

and

$$\tilde{x} - 1_\epsilon \equiv [\bar{x_s} - 1, \bar{x_f} - 1).$$

The rough points $\tilde{x} + 1_\epsilon$ and $\tilde{x} - 1_\epsilon$ are said to be **contiguous** to $\tilde{x}$. Quite similarly, $\tilde{x} + 2_\epsilon$, $\tilde{x} + 3_\epsilon$, etc. are defined.

2.1 Rough Point-Rough Point Relations

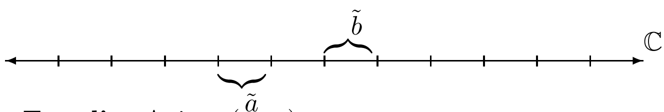
Three binary relations $\asymp_\eta, \prec_\eta, \succ_\eta$ may be defined on the tolerance approximation space $\mathbb{T}$ with a tolerance measure η ($\in \mathbb{R}^+$) [7]. Let $x, y \in \mathbb{R}$.

(*P*1) *Identity Axiom* $(\asymp_\eta)$:
$x \asymp_\eta y \Leftrightarrow (|x - y| < \eta)$
(*P*2) *Lesser Inequality Axiom* $(\prec_\eta)$:
$x \prec_\eta y \Leftrightarrow (x \leq y - \eta)$
(*P*3) *Greater Inequality Axiom* $(\succ_\eta)$:
$x \succ_\eta y \Leftrightarrow (x \geq y + \eta)$

In contrast, we observe the following five relations on the set of rough points defined in the communicative approximation space $\mathbb{C}$ with a granularity measure ϵ. Let $\tilde{a} \equiv [\ \bar{a_s},\ \bar{a_f})$ and $\tilde{b} \equiv [\ \bar{b_s},\ \bar{b_f})$ be two rough points.

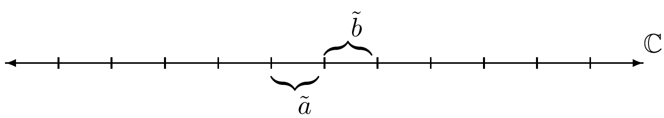
1. **Before** Axiom $(<_\epsilon)$:

$$\tilde{a} <_\epsilon \tilde{b} \Leftrightarrow (\bar{a_f} < \bar{b_s}) \qquad (\tilde{a}\ before\ \tilde{b})$$



2. **Before Equality** Axiom $(=_{<\epsilon})$:

$$\tilde{a} =_{<\epsilon} \tilde{b} \Leftrightarrow (\bar{a_f} = \bar{b_s}) \qquad (\tilde{a}\ equalsBefore\ \tilde{b})$$



3. **Exact Equality** Axiom $(=_{e\epsilon})$:

$$\tilde{a} =_{e\epsilon} \tilde{b} \Leftrightarrow (\bar{a_s} = \bar{b_s}) \qquad (\tilde{a}\ equalsExact\ \tilde{b})$$

The relations **After Equality**, and **After**, are defined in a dual manner.

Observation 2. *Let $\tilde{a} = [\bar{a_s}, \bar{a_f})$ and $\tilde{b} = [\bar{b_s}, \bar{b_f})$ be two rough points.*

1. *If $\tilde{a} <_\epsilon \tilde{b}$, there is an integer $k > 1$ such that $\tilde{b} = \tilde{a} + k_\epsilon$. If $\tilde{a} =_{<\epsilon} \tilde{b}$, $\tilde{b} = \tilde{a} + 1_\epsilon$, i.e. $\tilde{a}$, $\tilde{b}$ are contiguous.*
2. *$\tilde{a} <_\epsilon \tilde{b}$ if and only if there is a rough point $\tilde{c}$ such that $\tilde{a} <_\epsilon \tilde{c}$ and $\tilde{c} <_\epsilon \tilde{b}$, or $\tilde{a} =_{<\epsilon} \tilde{c}$ and $\tilde{c} =_{<\epsilon} \tilde{b}$, or $\tilde{a} <_\epsilon \tilde{c}$ and $\tilde{c} =_{<\epsilon} \tilde{b}$, or $\tilde{a} =_{<\epsilon} \tilde{c}$ and $\tilde{c} <_\epsilon \tilde{b}$.*
3. *The equalsExact relation is an equivalence relation on the set of rough points. It is, in fact, a congruence relation with respect to the equalsBefore and equalsAfter relations: $\tilde{a} =_{e\epsilon} \tilde{b}$ and $\tilde{b} R \tilde{c}$ imply $\tilde{a} R \tilde{c}$, where R is $=_{<\epsilon}$ or $=_{>\epsilon}$.*
4. *The before and after relations are transitive.*
5. *The relations in the pairs (before, after), and (equalsBefore, equalsAfter) are converses of each other: $\tilde{a} <_\epsilon \tilde{b} \Leftrightarrow \tilde{b} >_\epsilon \tilde{a}; \tilde{a} =_{<\epsilon} \tilde{b} \Leftrightarrow \tilde{b} =_{>\epsilon} \tilde{a}$.*

Remark 1. The correspondence between the set of all pairs of rough points and the set of all Rough Point-Rough Point relations defines a function.

Note 2. We shall drop the ϵ subscript in all our notations to make them more readable, but they would be assumed to be valid in some communicative approximation space $\mathbb{C}$ with a granularity ϵ.

Mapping Point-Point Relations in $\mathbb{C}$ and $\mathbb{T}$. As stated earlier, we consider a tolerance approximation space $\mathbb{T}$ with tolerance measure $\eta = \epsilon$. The transition from point-point relations in $\mathbb{T}$ to those in $\mathbb{C}$, and vice-versa is given by the following propositions. We drop the subscript ϵ in the notation of the $\mathbb{T}$-relations.

Proposition 1. $\mathbb{T}$ to $\mathbb{C}$: *For $x, y \in \mathbb{R}$, the $\mathbb{T}$-relations defined between them through $(P1)-(P3)$ are $\prec$, $\asymp$, and $\succ$. Then the possible $\mathbb{C}$-relations between the corresponding rough points $\tilde{x}, \tilde{y}$ are given in the table on the left.*

	$\mathbb{T}$-*Relation*	$\mathbb{C}$-*Relation*
(a)	$\prec$	$<$, $=_<$
(b)	$\asymp$	$=_<$, $=_e$, $=_>$
(c)	$\succ$	$=_>$, $>$

	$\mathbb{C}$-*Relation*	$\mathbb{T}$-*Relation*
(a)	$<$	$\prec$
(b)	$=_<$	$\asymp$, $\prec$
(c)	$=_e$	$\asymp$
(d)	$=_>$	$\asymp$, $\succ$
(e)	$>$	$\succ$

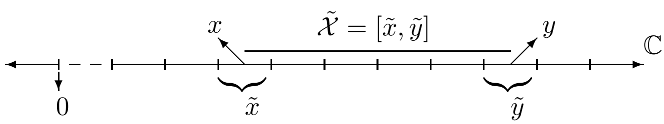
Proposition 2. $\mathbb{C}$ to $\mathbb{T}$: *Let $\tilde{x}$, $\tilde{y}$ be rough points. The possible $\mathbb{T}$-relations between any two real points $x \in \tilde{x}$, $y \in \tilde{y}$ are given in the table on the right.*

3 Rough Interval

Definition 2. *A* **rough interval** *is the union of any finite number of contiguous rough points $\tilde{x_1}, \tilde{x_2}, ..., \tilde{x_k}$.*

A rough point, in particular, is also a rough interval. Further, considering the rough interval $\tilde{x_1} \cup \tilde{x_2} \cup ... \cup \tilde{x_k}$, one observes that for any $x \in \tilde{x_1},\ y \in \tilde{x_k},\ \tilde{x_1} = \tilde{x},\ \tilde{x_2} = \tilde{x} + 1, ...,\ \tilde{x_k} = \tilde{y}$. In the terminology of rough set theory, for all such x, y, the intervals $X \equiv [x, y]$ are therefore *roughly equal.* They share the same upper approximation, which is the rough interval in question, and the same lower approximation (empty for $k = 1, 2$, and $\tilde{x_2} \cup \tilde{x_3} \cup ... \cup \tilde{x_{k-1}}$ for $(k \geq 3)$).

We denote the rough interval $\tilde{x_1} \cup \tilde{x_2} \cup ... \cup \tilde{x_k}$ as $\tilde{\mathcal{X}}$ – corresponding to any real interval $X \equiv [x, y]$ with x, y as above. Another denotation used would be $[\tilde{x},\ \tilde{y}]$, to indicate the 'starting rough point' and 'end rough point' of the rough interval.



Any real interval X is thus a *rough set* in the communicative approximation space $\mathbb{C}$, and $\tilde{\mathcal{X}}$ is its *upper approximation*. A rough interval, on the other hand, is a *definable/exact* set in $\mathbb{C}$. The *lower approximation* $\underline{X}$ of X in $\mathbb{C}$ is the rough interval $[\tilde{x}+1, \tilde{y}-1]$. It would also be termed the **interior** of the rough interval $\tilde{\mathcal{X}}$. If $\tilde{x}, \tilde{y}$ are contiguous or equal, $\underline{X}$ is empty.

In the degenerate case $x = y$, i.e. when $X = \{x\}$, $\tilde{\mathcal{X}}$ is just the rough point $\tilde{x}$. Generally, if $x < y$, we can have any of the three possibilities (i) $\tilde{x} < \tilde{y}$, (ii) $\tilde{x} =_< \tilde{y}$, or (iii) $\tilde{x} =_e \tilde{y}$. As noted in Observation 2, in case (i), $\tilde{y} = \tilde{x} + k$, for some integer k and so $\tilde{\mathcal{X}} = \tilde{x} \cup \tilde{x}+1 \cup \tilde{x}+2 \cup ... \cup \tilde{x}+k$. In case (ii), $\tilde{\mathcal{X}} = \tilde{x} \cup \tilde{x}+1$. (iii) gives $\tilde{\mathcal{X}} = \tilde{x}$ again.

Remark 2. Having said this, we observe that most discussions on interval algebras in tolerance spaces assume that $|I| \gg \eta$ for any real interval I, and tolerance measure η. For realistic discourse on a rough interval $\tilde{x_1} \cup \tilde{x_2} \cup ... \cup \tilde{x_k}$, we would expect that $k \gg 1$. Minimally, for a non-empty interior, we consider intervals with at least four contiguous rough points, i.e. we assume $\tilde{x_1} + 1 < \tilde{x_k}$.

3.1 Rough Point-Rough Interval Relations

Given a point x in $\mathbb{R}$, and a real interval I, we have the following binary relations between x and I in the tolerance approximation space $\mathbb{T}$ with a tolerance η:

$(PI1)$ $\quad x - I \Leftrightarrow x < i_1 - \eta$
$(PI2)$ $\quad x\ b\ I \Leftrightarrow x \asymp i_1 \Leftrightarrow |i_1 - x| < \eta$
$(PI3)$ $\quad x\ i\ I \Leftrightarrow x \in (i_1 + \eta, i_2 - \eta)$
$(PI4)$ $\quad x\ f\ I \Leftrightarrow x \asymp i_2 \Leftrightarrow |i_2 - x| < \eta$
$(PI5)$ $\quad x + I \Leftrightarrow x > i_2 + \eta$

In $\mathbb{C}$, a rough point $\tilde{x}$ and a rough interval $\tilde{\mathcal{I}} \equiv [\tilde{i}_1,\ \tilde{i}_2]$ have nine possible relations:

1. $\tilde{x}$ **before** $\tilde{\mathcal{I}}$ $(<)$: $\quad (\tilde{x} < \tilde{\mathcal{I}}) \Leftrightarrow (\tilde{x} < \tilde{i}_1)$
2. $\tilde{x}$ **startsBefore** $\tilde{\mathcal{I}}$ $(s_<)$: $\quad (\tilde{x}\ s_<\ \tilde{\mathcal{I}}) \Leftrightarrow (\tilde{x} =_< \tilde{i}_1)$
3. $\tilde{x}$ **startsExact** $\tilde{\mathcal{I}}$ (s_e): $\quad (\tilde{x}\ s_e\ \tilde{\mathcal{I}}) \Leftrightarrow (\tilde{x} =_e \tilde{i}_1)$
4. $\tilde{x}$ **startsAfter** $\tilde{\mathcal{I}}$ $(s_>)$: $\quad (\tilde{x}\ s_>\ \tilde{\mathcal{I}}) \Leftrightarrow (\tilde{x} =_> \tilde{i}_1)$
5. $\tilde{x}$ **interior** $\tilde{\mathcal{I}}$ (in): $\quad (\tilde{x}\ in\ \tilde{\mathcal{I}}) \Leftrightarrow (\tilde{i}_1 < \tilde{x} < \tilde{i}_2)$
   It may be remarked that the interior relation exists if and only if the interior of the rough interval has at least three contiguous rough points.
6. $\tilde{x}$ **finishesBefore** $\tilde{\mathcal{I}}$ $(f_<)$: $\quad (\tilde{x}\ f_<\ \tilde{\mathcal{I}}) \Leftrightarrow (\tilde{x} =_< \tilde{i}_2)$
7. $\tilde{x}$ **finishesExact** $\tilde{\mathcal{I}}$ (f_e): $\quad (\tilde{x}\ f_e\ \tilde{\mathcal{I}}) \Leftrightarrow (\tilde{x} =_e \tilde{i}_2)$
8. $\tilde{x}$ **finishesAfter** $\tilde{\mathcal{I}}$ $(f_>)$: $\quad (\tilde{x}\ f_>\ \tilde{\mathcal{I}}) \Leftrightarrow (\tilde{x} =_> \tilde{i}_2)$
9. $\tilde{x}$ **after** $\tilde{\mathcal{I}}$ $(>)$: $\quad (\tilde{x} > \tilde{\mathcal{I}}) \Leftrightarrow (\tilde{x} > \tilde{i}_2)$

Mapping Point-Interval Relations in $\mathbb{C}$ and $\mathbb{T}$. As in Section 2.1, we assume that the tolerance approximation space $\mathbb{T}$ has the tolerance measure $\eta = \epsilon$.

Proposition 3. $\mathbb{T}$ **to** $\mathbb{C}$**:** *Consider a real point x, a real interval $I \equiv [i_1, i_2]$, and the corresponding rough point $\tilde{x}$ and rough interval $\tilde{\mathcal{I}}$. The possible $\mathbb{C}$-relations between $\tilde{x}$ and $\tilde{\mathcal{I}}$ are given as follows.*

	$\mathbb{T}$-Relation	$\mathbb{C}$-Relation
(a)	$-$ $(x < i_1$ - $\epsilon)$	$<$, $s_<$
(b)	b $(\lvert i_1 - x\rvert < \epsilon)$	$s_<$, s_e, $s_>$
(c)	i $(x \in (i_1 + \epsilon, i_2 - \epsilon))$	$s_>$, in, $f_<$
(d)	f $(\lvert i_2 - x\rvert < \epsilon)$	$f_<$, f_e, $f_>$
(e)	$+$ $(x > i_2 + \epsilon)$	$f_>$, $>$

Proposition 4. $\mathbb{C}$ **to** $\mathbb{T}$**:** *Let $\tilde{x}$ be a rough point and $\tilde{\mathcal{I}} \equiv [\tilde{i_1}, \tilde{i_2}]$ a rough interval corresponding to any real interval I. The $\mathbb{C}$ to $\mathbb{T}$ mappings are unique, except for $\mathbb{C}$-Relations $s_<$ and $f_>$, for which the $\mathbb{T}$-Relations are $\{-, b\}$ and $\{f, +\}$ respectively.*

4 Relations between Rough Intervals

Relations between two rough intervals $\tilde{\mathcal{A}}$ $(\equiv [\tilde{a}_1, \tilde{a}_2])$ and $\tilde{\mathcal{B}}$ $(\equiv [\tilde{b}_1, \tilde{b}_2])$ are defined by the relations that the starting rough point $\tilde{a}_1$ and the end rough point $\tilde{a}_2$ of $\tilde{\mathcal{A}}$ have with the rough interval $\tilde{\mathcal{B}}$. Any such relation shall be represented by a pair (R_1, R_2), provided $\tilde{a}_1 R_1 \tilde{\mathcal{B}}$, and $\tilde{a}_2 R_2 \tilde{\mathcal{B}}$, where R_1, R_2 denote any of the relations defined in Section 3.1.
For example: $\tilde{\mathcal{A}}$ $(<, f_<)$ $\tilde{\mathcal{B}} \Leftrightarrow (\tilde{a}_1 < \tilde{\mathcal{B}})$ and $(\tilde{a}_2\ f_<\ \tilde{\mathcal{B}})$.

Remark 3. Due to the non-empty interior constraint, $(\tilde{a}_1 + 1_\epsilon < \tilde{a}_2) \wedge (\tilde{b}_1 + 1_\epsilon < \tilde{b}_2)$, some Rough Interval-Rough Interval Relations are not acceptable, e.g. $(\tilde{\mathcal{A}}(s_>, <)\tilde{\mathcal{B}})$ is not acceptable as $\tilde{a}_2 < \tilde{\mathcal{B}}$ and $\tilde{a}_1 < \tilde{a}_2 \Rightarrow \tilde{a}_1 < \tilde{\mathcal{B}}$ which contradicts $\tilde{\mathcal{A}}\ s_>\ \tilde{\mathcal{B}}$.

So, we have the following possible relations between two rough intervals $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ – given in the Table 1.

Observation 3. *Inclusion:* Considering ordinary set inclusion, we have

1. $\tilde{\mathcal{A}} \subset \tilde{\mathcal{B}} \Leftrightarrow \tilde{\mathcal{A}}$ $(=_{e<}, =_{><}, =_{>e}, s_e, s_>, cb, f_<, f_e)$ $\tilde{\mathcal{B}}$, *and*
2. $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}} \Leftrightarrow \tilde{\mathcal{A}}$ $(=_{e<}, =_{ee}, =_{><}, =_{>e}, s_e, s_>, cb, f_<, f_e)$ $\tilde{\mathcal{B}}$.

One may define *containment* $(\supset / \supseteq)$ dually.

4.1 Mapping Interval-Interval Relations from Toleranced Real Model

Proposition 5. *Interval-interval relations are written concatenated from the point-interval relations: thus the notation A bf B indicates that interval A begins (b) and finishes (f) at the same points as B, i.e., the intervals A and B are equal. There are 13 relations between two real intervals in the Toleranced Real Model [7]. The Rough Set Model relations for corresponding rough intervals are given below for seven relations (the other six are inverses of cases a-f).*

Table 1. Relations between two rough intervals $\tilde{\mathcal{A}}[\tilde{a}_1, \tilde{a}_2]$ and $\tilde{\mathcal{B}}[\tilde{b}_1, \tilde{b}_2]$; 19 (including inverses) of the 33 relations are shown; the others are finishedBy{After,Exact,Before}, contains, starts{Before,Exact,After} which have inverses in finishes{After,Exact,Before}, containedBy, startedBy{Before,Exact,After} respectively. (Yes, do let us know if you can suggest some more readable names!)

$\tilde{\mathcal{A}}$ Relation $\tilde{\mathcal{B}}$	Definition	$\tilde{\mathcal{B}}$ Relation $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ before $\tilde{\mathcal{B}}$ $(<)$	$(\tilde{\mathcal{A}} < \tilde{\mathcal{B}}) \Leftrightarrow (\tilde{a}_2 < \tilde{\mathcal{B}})$	$\tilde{\mathcal{B}}$ after $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ meetsBefore $\tilde{\mathcal{B}}(m_<)$	$(\tilde{\mathcal{A}}\ m_<\ \tilde{\mathcal{B}}) \Leftrightarrow (\tilde{a}_2\ s_<\ \tilde{\mathcal{B}})$	$\tilde{\mathcal{B}}$ metByBefore $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ meetsExact $\tilde{\mathcal{B}}(m_e)$	$(\tilde{\mathcal{A}}\ m_e\ \tilde{\mathcal{B}}) \Leftrightarrow (\tilde{a}_2\ s_e\ \tilde{\mathcal{B}})$	$\tilde{\mathcal{B}}$ metByExact $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ meetsAfter $\tilde{\mathcal{B}}$ $(m_>)$	$(\tilde{\mathcal{A}}\ m_> \tilde{\mathcal{B}}) \Leftrightarrow (\tilde{a}_2\ s_> \tilde{\mathcal{B}})$	$\tilde{\mathcal{B}}$ metByAfter $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ overlaps $\tilde{\mathcal{B}}$ (o)	$(\tilde{\mathcal{A}}\ o\ \tilde{\mathcal{B}}) \Leftrightarrow ((\tilde{a}_1 < \tilde{\mathcal{B}}) \wedge (\tilde{a}_2\ i\ \tilde{\mathcal{B}}))$	$\tilde{\mathcal{B}}$ overlappedBy $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ equalsBeforeBefore $\tilde{\mathcal{B}}$ $(=_{<<})$	$(\tilde{\mathcal{A}} =_{<<} \tilde{\mathcal{B}}) \Leftrightarrow ((\tilde{a}_1\ s_<\ \tilde{\mathcal{B}}) \wedge (\tilde{a}_2\ f_<\ \tilde{\mathcal{B}}))$	$\tilde{\mathcal{B}}$ equalsAfterAfter $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ equalsBeforeExact $\tilde{\mathcal{B}}(=_{<e})$	$(\tilde{\mathcal{A}} =_{<e} \tilde{\mathcal{B}}) \Leftrightarrow ((\tilde{a}_1\ s_<\ \tilde{\mathcal{B}}) \wedge (\tilde{a}_2\ f_e\ \tilde{\mathcal{B}}))$	$\tilde{\mathcal{B}}$ equalsAfterExact $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ equalsBeforeAfter $\tilde{\mathcal{B}}$ $(=_{<>})$	$(\tilde{\mathcal{A}} =_{<>} \tilde{\mathcal{B}}) \Leftrightarrow ((\tilde{a}_1\ s_<\ \tilde{\mathcal{B}}) \wedge (\tilde{a}_2\ f_>\ \tilde{\mathcal{B}}))$	$\tilde{\mathcal{B}}$ equalsAfterBefore $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ equalsExactBefore $\tilde{\mathcal{B}}$ $(=_{e<})$	$(\tilde{\mathcal{A}} =_{e<} \tilde{\mathcal{B}}) \Leftrightarrow ((\tilde{a}_1\ s_e\ \tilde{\mathcal{B}}) \wedge (\tilde{a}_2\ f_<\ \tilde{\mathcal{B}}))$	$\tilde{\mathcal{B}}$ equalsExactAfter $\tilde{\mathcal{A}}$
$\tilde{\mathcal{A}}$ equalsExactExact $\tilde{\mathcal{B}}(=_{ee})$:	$(\tilde{\mathcal{A}} =_{ee} \tilde{\mathcal{B}}) \Leftrightarrow ((\tilde{a}_1\ s_e\ \tilde{\mathcal{B}}) \wedge (\tilde{a}_2\ f_e\ \tilde{\mathcal{B}}))$	$\tilde{\mathcal{B}}$ equalsExactExact $\tilde{\mathcal{A}}$

	Tolerance Model Relation	*Rough Set Model Relations*
(a)	$--$ (before)	$<$, $m_<$
(b)	$-b$ (meets)	$m_<$, m_e, $m_>$
(c)	$-i$ (overlaps)	$m_>$, o, $fb_>$, $s_<$, $=_{<<}$
(d)	$-f$ (finishedBy)	$fb_>$, fb_e, $fb_<$, $=_{<<}$, $=_{<e}$, $=_{<>}$
(e)	$-f\ -+$ (contains)	$fb_<$, c, $=_{<>}$, $sb_>$
(f)	$-f\ bi$ (starts)	$s_<$, s_e, $s_>$, $=_{<<}$, $=_{e<}$, $=_{><}$
(g)	$-f\ bf$ (equals)	any one of all 9 equalities ($=_{<<}$, $=_{<e}$, etc.)

From Rough Set Model to Toleranced Real Model. Two rough intervals can have 33 Rough Set Model relations among them. The corresponding relations in Toleranced Real Model for the corresponding real points and real intervals can be determined as in the earlier cases, but are omitted here.

5 Algebraic Aspects: A Preliminary Study

Rough sets have been extensively studied from the algebraic viewpoint (cf. [3]). In particular, a study in the context of *relation algebras* [6] may be found, for instance, in the work of Düntsch [4] where, following Tarski, a generalized notion of a 'rough relation algebra' is defined. Our interest here is slightly different. It is well-known that points and intervals on the rationals or reals constitute basic examples of relation algebras. We investigate the relational algebraic structure obtained from the rough points defined here.

Let us consider the communicative space $\mathbb{C}$ and the field $\mathcal{R}(\mathbb{C})$ of binary relations over $\mathbb{R}/\epsilon$ (the collection of all rough points in $\mathbb{C}$), i.e.

$$\mathcal{R}(\mathbb{C}) \equiv (\mathcal{P}(\mathbb{R}/\epsilon \times \mathbb{R}/\epsilon), \cup, ^c, \emptyset, \mathbb{R}/\epsilon \times \mathbb{R}/\epsilon, ^\smile, ;, 1'),$$

where $\mathcal{P}$ represents the power set, $\smile$ is the converse operation, ; the composition operation and $1'$ the identity relation.

Now let us look at the set of five relations between rough points on $\mathbb{C}$ (cf. Section 2.1), $X_0 \equiv \{<, =_<, =_e, =_>, >\}$, and the subalgebra $\mathcal{S}(X_0)$ of $\mathcal{R}(\mathbb{C})$ generated by X_0. As noted in Observation 2(5), $<$ is $>^{\smile}$, and $=_<$ is $=_>^{\smile}$. In the following proposition, we mention some results of composition of the relations in X_0. For any relation R, let $R\, ;^n R$ denote $R; R; \ldots ; R$ (n times).

Proposition 6

1. $<;^{n+1} < \;\subset\; <;^n < \;\subset\; \ldots \;\subset\; <;< \;\subset\; <.$
2. $<;^n < \;=\; <;(=_<;^{2n-1} =_<).$
3. $=_<;^n =_< \;\subset\; <.$
   However, $=_<;^n =_<$ *is not comparable with* $=_<;^m =_<$*, where* $m \neq n$.

Using Observation 1, we see that the set of relations obtained by the composition and converse operations on elements of X_0 is isomorphic to the set $\mathcal{Z}$ of integers. Moreover, from Proposition 6 we conclude that the subalgebra $\mathcal{S}(X_0)$ is infinite. Further, closure with respect to $\cup$, $\cap$, and complementation (to make a Boolean algebra) gives that $\mathcal{S}(X_0)$ is isomorphic to the Boolean algebra of all finite and co-finite subsets of $\mathcal{Z}$. Thus, $\mathcal{S}(X_0)$ is isomorphic to a subalgebra of the complex algebra [6] $(\mathcal{P}(\mathcal{Z}), \cup, ^c, ^\smile, ;, 1')$ of the group $(\mathcal{Z}, +, 0)$.[1]

6 Conclusion

In this paper we present an approach for mapping quantities in a communicative approximation space where indistinguishability relations are modeled through rough intervals. In this work, we assume that intervals exhibit similar tolerances at both end points; where this does not hold, one needs to construct a formalism for asymmetric relations. The rough interval formalism introduced here is aimed merely at capturing the communication tolerances where explicit quantities are mentioned, how such tolerances are to be identified remains a complex question in pragmatics, and is beyond the scope of the present work.

A preliminary study of the relational algebraic aspects of the constructs defined here, has been reported in this article. Much more needs to be investigated, for instance, structures that are formed by rough intervals. In any case, it is clear that we shall obtain an infinite relation algebra for rough intervals as well.

Finally, we have assumed that the communicative tolerance ϵ is about the same as the observation tolerance $\pm\tau$. However, sometimes these two may be quite disparate – e.g. we may read that the time is "9:23.43", but we may not use such an accuracy in reporting it if we know that the listener has no use for such precision. Thus, situations with asymmetric observational and communicational tolerance also deserve further analysis, which has not been attempted here.

[1] Discussions with Robin Hirsch and Ian Hodkinson helped in relating our algebra to the group relation algebra over $\mathcal{Z}$.

Another important aspect is that of making transitive inferences between communicatively specified events. There is a large literature on complexity classes associated with transitive inference; for interval algebras defined on the real line, subalgebras involving contiguous relations are usually found to be tractable, whereas the full algebras are generally NP-hard [13]. We suspect this may also be the case for transitivity here, but this requires formal verification.

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