Matrix Factorization and Matrix Completion

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Machine Learning (CS771A)

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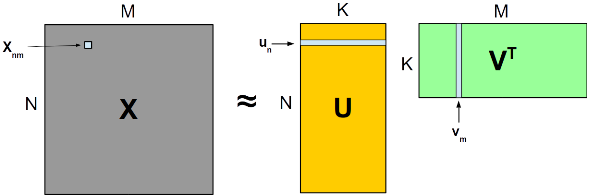
Matrix Factorization

- Given a matrix $\mathbf{X}$ of size $N \times M$, approximate it as a product of two matrices

$$\mathbf{X} \approx \mathbf{U}\mathbf{V}^\top$$



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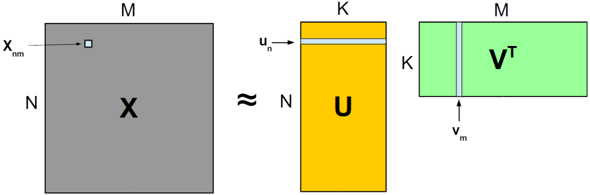

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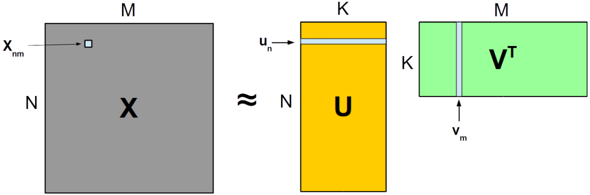

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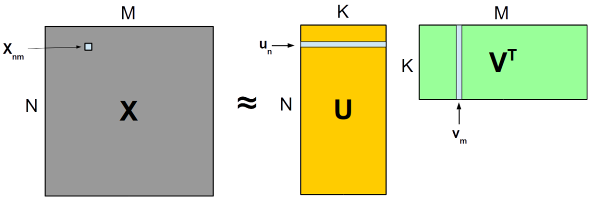

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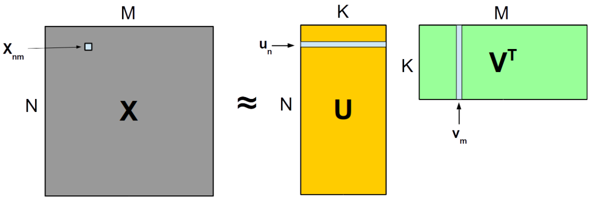

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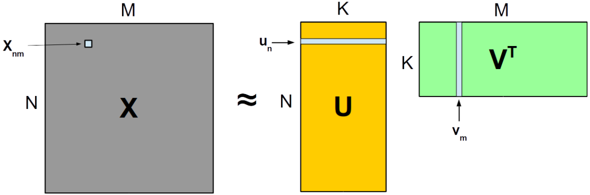

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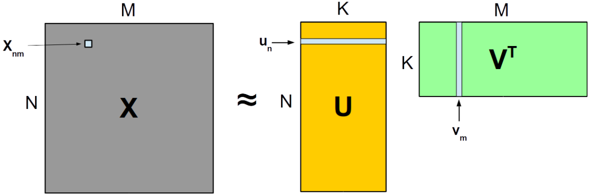

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- If X_{nm} is large (small) then $\boldsymbol{u}_n$ and $\boldsymbol{v}_m$ should be similar (dissimilar)

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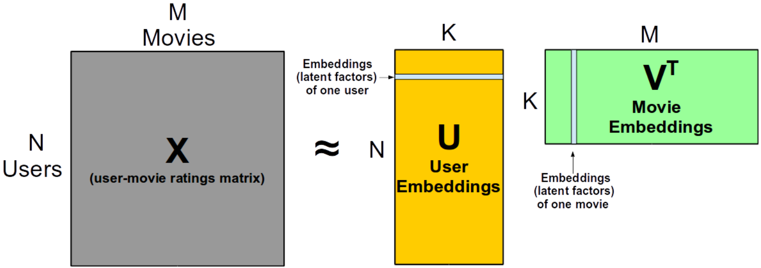
- The latent factors can be used/interpreted as "embeddings" or "features"



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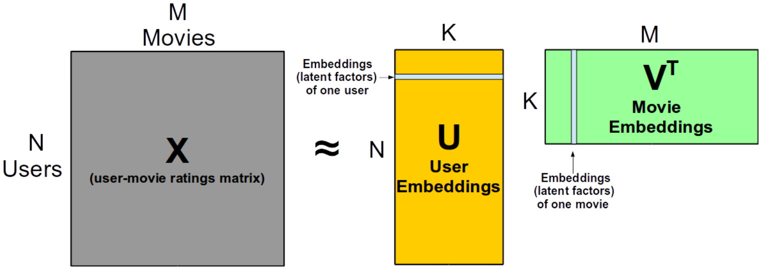

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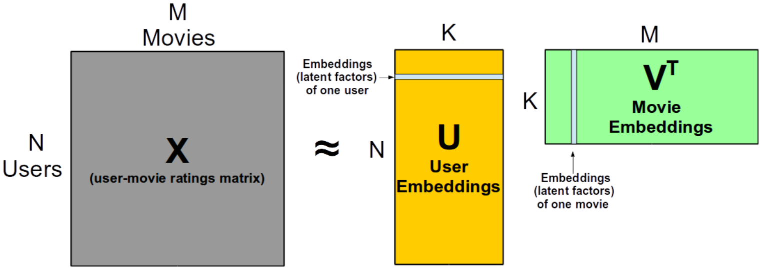

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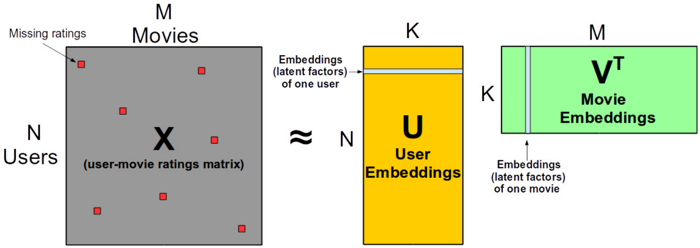
- If $K \ll \min\{M, N\} \Rightarrow$ then can also be seen as dimensionality reduction or a "low-rank factorization" of the matrix $\mathbf{X}$

Why Matrix Factorization?

- Can also predict the missing/unknown entries in the original matrix



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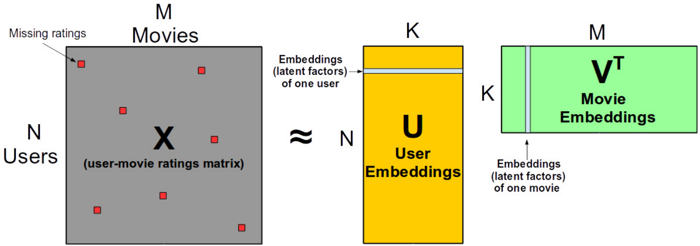

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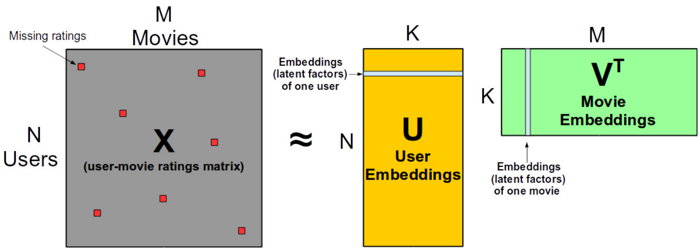

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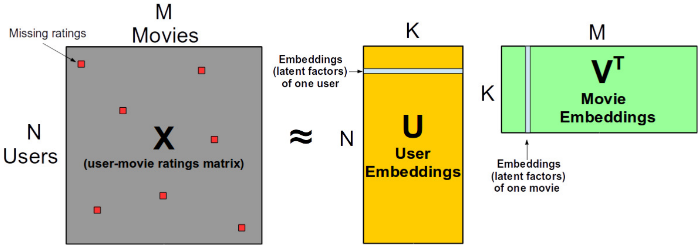

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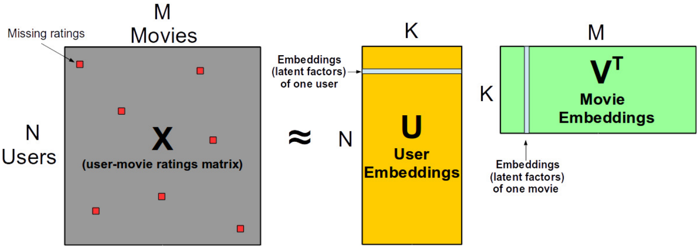

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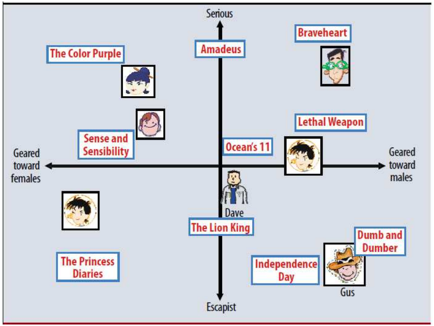
- Note: The "Netflix Challenge" was won by a matrix factorization method

Interpreting the Embeddings/Latent Factors

- Embeddings/latent factors can often be interpreted. E.g., as “genres” if $\mathbf{X}$ represents a user-movie rating matrix. A cartoon with $K = 2$ shown below



Picture courtesy: Matrix Factorization Techniques for Recommender Systems: Koren *et al*, 2009

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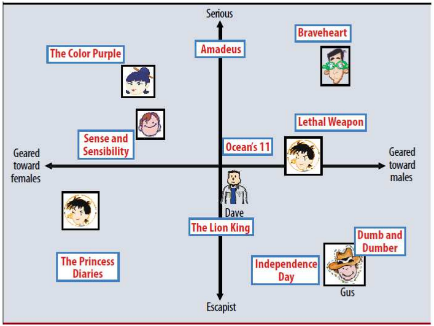

- Similar things (users/movies) get embedded nearby in the embedding space (two things will be deemed similar if their embeddings are similar). Thus useful for computing similarities and/or making recommendations

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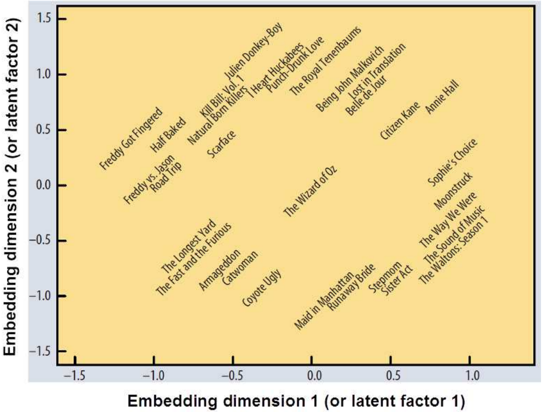
Interpreting the Embeddings/Latent Factors

- Another illustation of 2-D embeddings of the movies only



- Similar movies will be embedded at nearby locations in the embedding space

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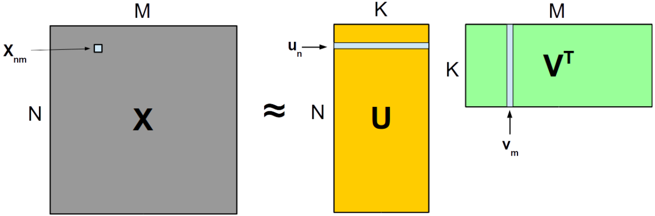
Solving Matrix Factorization

Matrix Factorization

- Recall our matrix factorization model: $\mathbf{X} \approx \mathbf{U}\mathbf{V}^\top$
- Goal: learn $\mathbf{U}$ and $\mathbf{V}$, given a subset Ω of entries in $\mathbf{X}$ (let's call it $\mathbf{X}_\Omega$)
- Some notations:
  - $\Omega = \{(n, m)\}$: X_{nm} is observed
  - Ω_{r_n}: column indices of observed entries in row n of $\mathbf{X}$
  - Ω_{c_m}: row indices of observed entries in column m of $\mathbf{X}$



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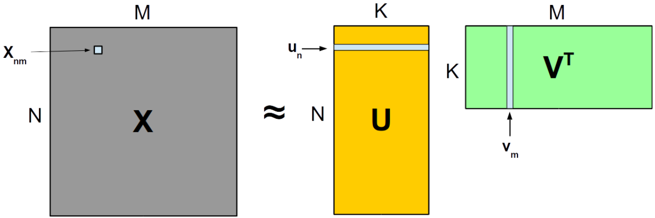
- We want $\mathbf{X}$ to be as close to $\mathbf{U}\mathbf{V}^\top$ as possible



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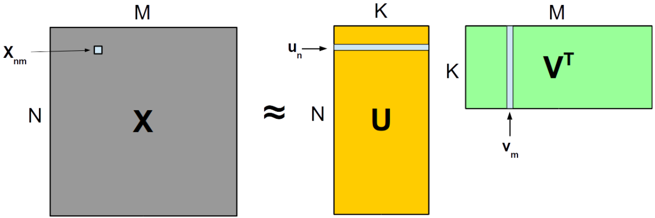

- Let's define a squared "loss function" over the observed entries in $\mathbf{X}$

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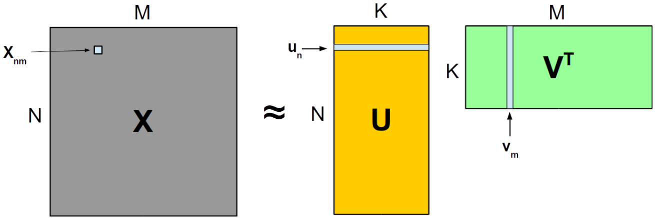

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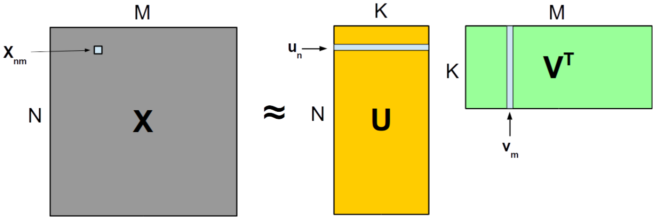

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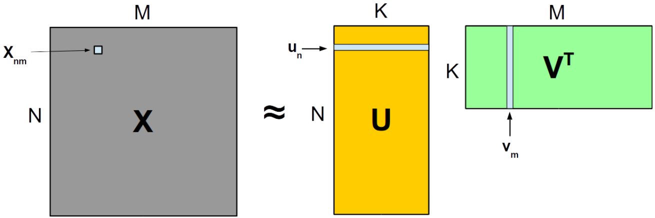

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- How do we learn $\{\boldsymbol{u}_n\}_{n=1}^N$ and $\{\boldsymbol{v}_m\}_{m=1}^M$?

Alternating Optimization

- We will use an ℓ_2 regularized version of the squared loss function

$$\mathcal{L} = \sum_{(n,m)\in\Omega} (X_{nm} - \boldsymbol{u}_n^\top \boldsymbol{v}_m)^2 + \sum_{n=1}^{N} \lambda_U ||\boldsymbol{u}_n||^2 + \sum_{m=1}^{M} \lambda_V ||\boldsymbol{v}_m||^2$$

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- Convergence properties of such methods have been studied extensively

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The Solutions

- Easy to show that the problem

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- Note that this is very similar to (regularized) least squares regression
- Thus matrix factorization can be also seen as a sequence of regression problems (one for each latent factor)

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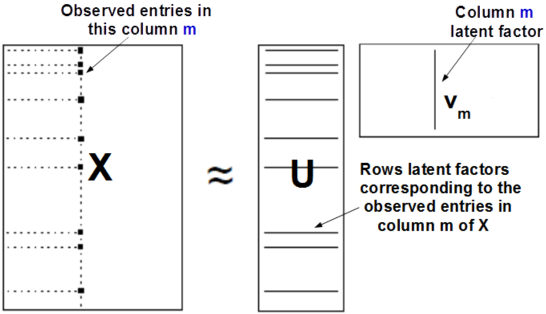

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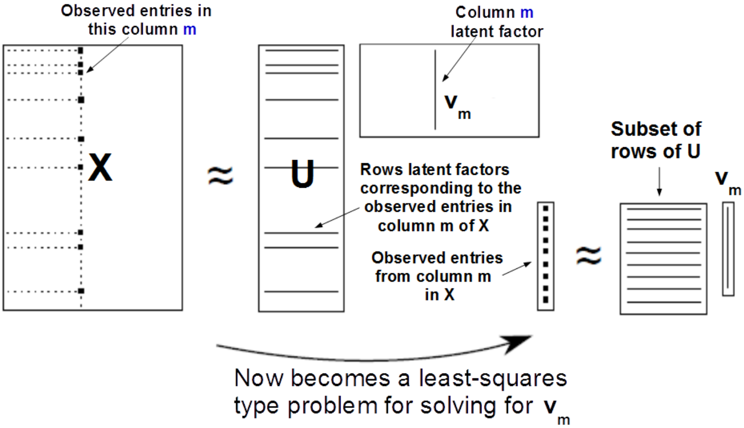

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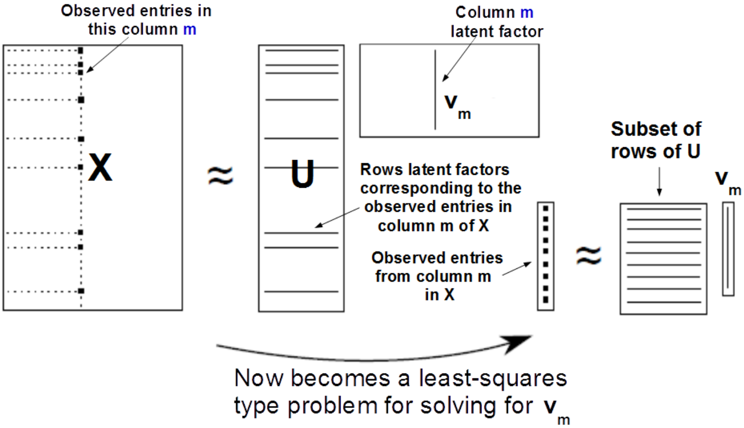

Can think of solving for $\boldsymbol{u}_n$ (with $\mathbf{V}$ and all other $\boldsymbol{u}_n$'s fixed) in the same way

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  - Can think of this also as a probabilistic model (a likelihood function on X_{nm} and priors on the latent factors $\boldsymbol{u}_n$, $\boldsymbol{v}_m$) and do MLE/MAP

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- Final prediction for any (missing) entry: $X_{nm} = \boldsymbol{u}_n^\top \boldsymbol{v}_m$

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- The SGD algorithm chooses a random entry X_{nm} in each iteration, updates $\boldsymbol{u}_n, \boldsymbol{v}_m$, and repeats until convergece ($\boldsymbol{u}_n$'s,$\boldsymbol{v}_m$'s randomly initialized).

Some Other Extensions of Matrix Factorization

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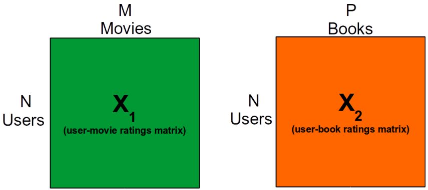

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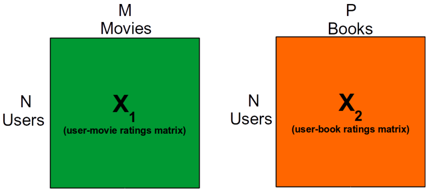

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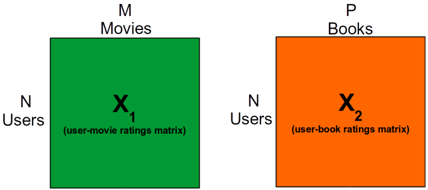

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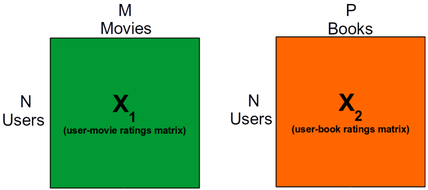

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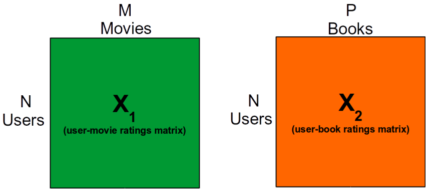

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Tensor Factorization

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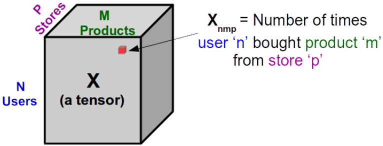
- Consider a 3-dim (or 3-mode or 3-way) tensor $\mathbf{X}$ of size $N \times M \times P$



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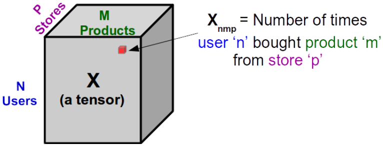

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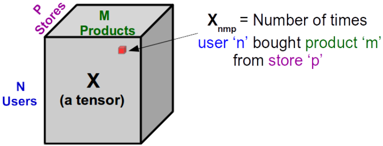

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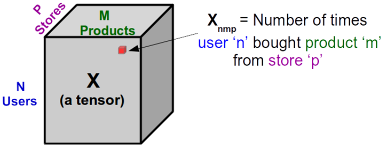

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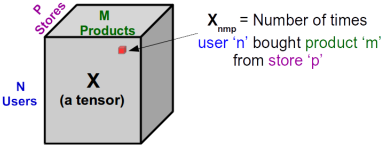

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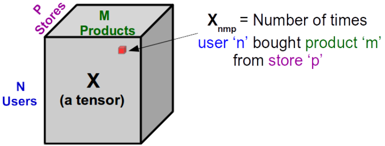

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- Several specialized algorithms for tensor factorization (CP/Tucker decomposition, etc.)

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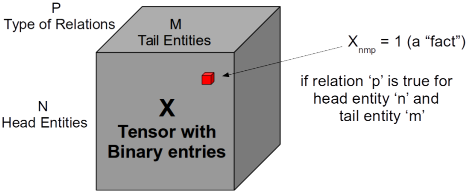

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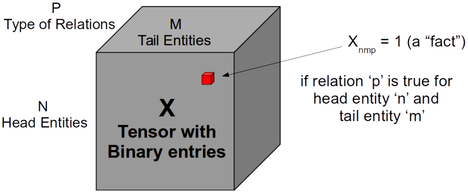

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- We can applying tensor factorization to learn features/embeddings of the entities and relations. Can use the embeddings to predict the unknown facts

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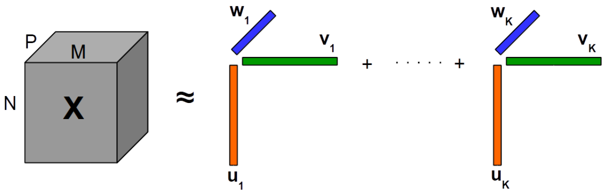
- E.g., an $N \times M \times P$ tensor $\mathbf{X}$ as a sum of outer products of column vectors

$$\mathbf{X} \approx \sum_{k=1}^{K} \boldsymbol{u}_k \otimes \boldsymbol{v}_k \otimes \boldsymbol{w}_k \qquad \text{(tensor SVD view; also generalizes to more than 3 dims)}$$

$\boldsymbol{u}_k \in \mathbb{R}^N$, $\boldsymbol{v}_k \in \mathbb{R}^M$ and $\boldsymbol{w}_k \in \mathbb{R}^P$ denote the k-th columns of $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ resp.



Matrix/Tensor Factorization: Another View

- We modeled each matrix/tensor entry as an inner product of latent factors
- Can also model matrix (or tensor) as a whole, as sum of rank-1 components
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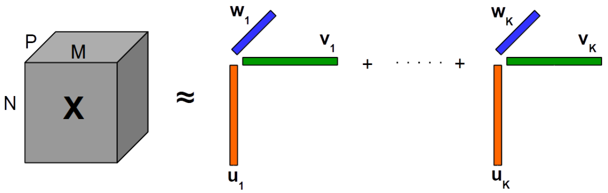

- The matrix case is similar (only 2 dimensions)

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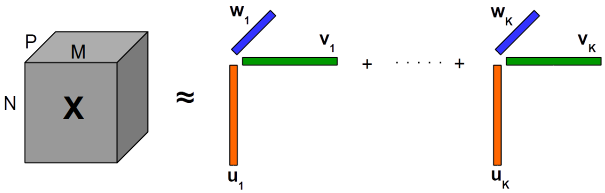

- The matrix case is similar (only 2 dimensions)
- With this view, in alternating optimization, can update one column at a time